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ON CERTAIN CLASSES OF FIRST BAIRE FUNCTIONS

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Chapter 1

Introduction

A real valued function is Baire-1 if pointwise limit of a sequence of continuous functions. This particular class of functions was introduced by Baire in his doctoral thesis at at École Normale Supérieure in 1899 and from that moment it caught the interest of several mathematicians.

H. Rosenthal was one of the first to study topological properties of the set of all Baire-1 functions on a given Polish space, endowed with the point-wise topology.

After that he worked together with E. Odell, discovering that the existence of Baire-1 members in the bidual of a separable Banach space is equivalent to prove that that space contains an isomorphic copy of ℓ_1.

Even the Fields medal Jean Bourgain, focused part of his youth works on the study of this classes of functions, improving the Theorems of Rosenthal and Odell.

He also introduced two subclasses of the set of all Baire-1 functions on a separable Banach space, with the purpose to deepen the connection with ℓ_1.

The last part of this work is based on an original result of the author and his supervisor, who studied the descriptive set nature of the two subclasses introduced by Bourgain.

1.1 Some basic facts

Definition 1.1. A topological space is said to be **Polish** if it is separable and completely metrizable.

If X is a normed space, we denote by $B_X = \{x \in X : \|x\|_X \leq 1\}$ the unit ball of X.

The **dual** of X is the space of all bounded linear functionals on X, with the operator norm and it is denoted by X^*. Analogously the **bidual** of X is defined as the dual of X^*.

Remark 1.2. The dual of a normed space is always a Banach space, endowed with the norm $\|f\|_{X^*} = \sup\{|f(x)| : x \in B_X\}$.

Remark 1.3. Let X be a normed space. The map $J : X \to X^{**}$ which associate an element $x \in X$ to the map

$$\begin{aligned} J(x) :& X^* \to \mathbb{R} \\ & \varphi \mapsto \varphi(x) \end{aligned}$$

Is an injective isomorphism, and it is known as the **canonical embedding** of X into X^{**}.

Throughout this work we will often consider X as a subspace of X^{**} via the canonical embedding, even if not explicitly indicated.

Definition 1.4. Let us recal that

- the **weak topology** on X is the weakest topology making all maps of X^* continuous;
- the **weak* topology** on X^* is the weakest topology making all maps of $J(X) \subseteq X^{**}$ continuous.

Remark 1.5.

1. $(x_n)_{n\in\mathbb{N}}$ in X converges weakly to $x \in X$ if and only if $(\varphi(x)_n)_{n\in\mathbb{N}}$ converges to $\varphi(x)$ for all $\varphi \in X^*$;
2. $(\varphi_n)_{n\in\mathbb{N}}$ in X^* weak*-converges to $\varphi \in X^*$ if and only if $(\varphi_n)_{n\in\mathbb{N}}$ pointwise-converges to φ, i.e. $\lim_n \varphi_n(x) = \varphi(x)$ for all $x \in X$.

Let us recall that

$$\ell_1 = \left\{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} : \sum_{i\in\mathbb{N}} |x_n| < \infty\right\}.$$

is a Banach space with the norm $\|x\|_{\ell_1} = \sum_{i\in\mathbb{N}} |x_n|$.

Remark 1.6. No element of $\ell_1^{**} \setminus \ell_1$ is a weak* limit of a sequence of elements of ℓ_1.

Definition 1.7. If X is a topological space, denote by $C(X)$ the set of all continuous real valued functions defined on X.

Remark 1.8. If X is a compact topological space, then $C(X)$ with the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

is a Banach space.

Let us denote by ω the first infinite ordinal, which corresponds to the sets of all non-negative integers. If A is a non-empty set, denote by

$$A^\omega = \prod_{n \in \omega} A = \{(a_n)_{n \in \mathbb{N}} \subset A\}$$

If we see A with the discrete topology (i.e. every subset of A is open), we can endow A^ω with the product topology. Such topology is induced by the metric $d(x, y) = \frac{1}{2^{n+1}}$ where $n \in \omega$ is the first positive integer such that $x(n) \neq y(n)$.

With the previous notations, the set $2^\omega = 0, 1^\omega$ is called the **Cantor space**. Remark that the map

$$\begin{aligned} \Phi \colon 2^\omega &\to \mathbb{R} \\ (a_n)_n &\mapsto \sum_n \frac{2a_n}{3^{n+1}} \end{aligned}$$

gives a homeomorphism from 2^ω onto the usual Cantor space of $[0, 1]$.

We now present two well-known Theorems in Functional Analysis. For details about their proofs, we refer the reader the book [Meg98].

Theorem 1.9 (Banach-Alaoglu-Bourbaki)**.** *If X is a normed space, then B_{X^*} is w^*-compact.*

Theorem 1.10 (Goldstine)**.** *If X is a normed space, then $J(B_X)$ is weakly* dense in $B_{X^{**}}$.*

Chapter 2

Baire-1 function

Baire functions are functions obtained by transfinite iteration of the operation of forming pointwise limits of sequences of continuous functions. They were introduced by René-Louis Baire in 1899 in his doctoral thesis at École Normale Supérieure ([Bai99]). Other properties of this class of functions are also described in his book "Leçons sur les fonctions discontinues, professées au collège de France" ([Bai05]).

The first part of this chapter is a short introduction on Baire-1 functions. Afterwards we present an article by Rosenthal ([Ros77]), who worked on some topological properties of the set of all Baire-1 functions on a Polish space endowed with the pointwise topology.

To conclude we illustrate the work [OR75] by Odell and Rosenthal, who found out a strict connection in separable Banach spaces, between containing an isomorphic copy of ℓ_1 and having Baire-1 members in the bidual.

2.1 Baire class 1 functions

Definition 2.1. Let (X, d) be a complete metric space, a function $f\colon X \to \mathbb{R}$ is said to be **Baire class 1** or simply **Baire-1** if it is the pointwise limit of a sequence of continuous functions $(f_n)_{n\in\mathbb{N}}\colon X \to \mathbb{R}$.

We denote by $\underline{\mathcal{B}}_1(X)$ the set of all Baire-1 functions on X.

It is clear that every continuous function is Baire-1, however the converse is not true. An example of Baire-1 function which is not continuous is the pointwise limit of the sequence $f_n(x) = x^n$ in $[0, 1]$.

As the name suggests, there is a strong connection between Baire-1 functions and Baire set category. Let us recall some basic definitions.

Definition 2.2. Let X be a topological space. A subset of X is said to be:

- **nowhere dense** if $\mathrm{int}(\overline{A}) = \emptyset$;
- **meagre** or **Baire first category** if it is a countable union of nowhere dense sets;
- **Baire second category** if it is not meagre.

Definition 2.3. A topological space X is said to be a **Baire space** if each open subsets of X is Baire second category.

Let us recall two well known Theorems by Baire. Their proofs can be found in [Bai99].

Theorem 2.4 (Baire Category)**.** *Let X be a topological space. X is a Baire space if and only if for all sequences of open dense subset of X $(A_n)_{n\in\mathbb{N}}$ then $\bigcap_{n\in|N} A_n$ is an open dense subset of X.*

Theorem 2.5. *Every complete metric space is a Baire space.*

The following Theorem explains the reason of name of this class of functions.

Theorem 2.6. *Let (X, d) be a complete metric space, let $f\colon X \to \mathbb{R}$ a Baire-1 function. Then the set of points of discontinuity of f is Baire first category.*

Actually, in [Bai05] Baire proved a stronger version of the previous Theorem.

Theorem 2.7 (Baire characterization Theorem)**.** *Let $f\colon X \longrightarrow \mathbb{R}$. Then f belongs to the first Baire class on X if and only if for every non-empty closed subset M of X, $f|_M$ has a point of continuity relative to the topological space M.*

2.2 Point-wise compact subsets of $\underline{\mathcal{B}}_1(X)$

We now begin the description of the article [Ros77] by Rosenthal.
In his work he introduced a class of badly discontinous functions, which play a key role in the proofs of the Theorems in this section.

Definition 2.8. Let Y be a topological space and $f\colon Y \longrightarrow \mathbb{R}$. We say that f satisfies the **Discontinuity Criterion** provided there is a non-empty subset L of Y and $r, \delta \in \mathbb{R}$ with $\delta > 0$ so that

$$\begin{aligned}&\text{for every non empty relatively open subset } U \text{ of } L,\\ &\text{there are } y, z \in U \text{ with } f(y) > r + \delta \quad\text{and}\quad f(z) < r\end{aligned} \tag{2.1}$$

Definition 2.9. Let A, B be two infinite subsets of $\mathbb{N}$. We say that A is **almost contained** in B, and write $A \subset_a B$, when $A \cap (\mathbb{N} \setminus B)$ is finite.

Notice that if $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sets, it is easy to find an infinite set B such that $B \subset_a A_n$ for all $n \in \mathbb{N}$. For instance, it is sufficient to choose one element from each A_n.

Lemma 2.10. *Let X be a Polish space, let $(f_n)_{n\in\mathbb{N}}\colon X \longrightarrow \mathbb{R}$ be a pointwise bounded sequence of functions that has no pointwise convergent subsequence. Then there are $N \subset \mathbb{N}$ and $r, \delta \in \mathbb{R}$ with $\delta > 0$ such that for every infinite subset $M \subseteq N$ there is an $x \in X$ satisfying*

$$\begin{array}{ll} f_m(x) > r + \delta & \textit{for infinitely many } m \in M \textit{ and} \\ f_m(x) < r & \textit{for infinitely many } m \in M \end{array} \tag{2.2}$$

Proof. Let $(r_n, \delta_n)_{n\in\mathbb{N}}$ be a enumeration of $\mathbb{Q} \times \mathbb{Q}^+$. Suppose that the thesis of Lemma 2.10 is false, and choose $M_0 = \mathbb{N}$. Then there is an infinite subset M_1 of M_0 such that every $x \in X$ fails to satisfy (2.2) with $r = r_0$ and $\delta = \delta_0$. By induction, for all $n \in \mathbb{N}$ we can build an infinite subset M_n of M_{n-1} such that every $x \in X$ fails to satisfy (2.2) with $M = M_n$, $r = r_{n-1}$ and $\delta = \delta_{n-1}$. So we built a decreasing sequence $(M_n)_{n\in\mathbb{N}}$ of infinite subset of $\mathbb{N}$, thus by the previous observation, there exists a set M such that $M \subset_a M_n$ for all $n \in \mathbb{N}$. Therefore for every $x \in X$, there is no pair (r_n, δ_n) satisfying (2.2).
On the other hand, by assumption $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded and has no pointwise convergent subsequences, thus there is an $x \in X$ such that

$$\liminf_{n\in M} f_n(x) < \limsup_{n\in M} f_n(x)$$

Now simply choose r and $\delta > 0$ such that

$$\liminf_{n\in M} f_n(x) < r < r + \delta < \limsup_{n\in M} f_n(x)$$

Hence x satisfies (2.2), which is a contradiction. □

Let us recall that a topological space is said **second-countable** if it has a countable base of open sets. Remember that a separable metrizable space is always second-countable.

Remark 2.11. If X is a second-countable topological space, then any strictly monotone net $(U_\alpha)_{\alpha<\gamma}$ of open subsets of X has countable length, i.e. $\gamma \leq \omega_0$.

Proof. Let $(B_n)_{n\in\mathbb{N}}$ a countable base for X. By definition we can write each U_α as union (at most countable) of elements of the base. By strict monotony, for each $\alpha < \gamma$ there is a $B_{n_\alpha} \subseteq U_{\alpha+1} \setminus U_\alpha$ and of course all such B_{n_α} are pairwise distinct. But for this to be true it should be $\gamma \leq \omega_0$. □

Theorem 2.12. *Let $(f_n)_{n\in\mathbb{N}} \colon X \longrightarrow \mathbb{R}$ be a pointwise bounded sequence of functions that has no pointwise convergent subsequence. Then there exists a non-empty subset L of X and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ pointwise convergent on L towards a function f which satisfies the Discontinuity Criterion.*

Proof. By Lemma 2.10 there are $N \subset \mathbb{N}$ and $r, \delta \in \mathbb{R}$ with $\delta > 0$ such that (2.2) holds. For every infinite $M \subseteq N$ let us define

$$K(M) = \overline{\{x \in X \text{ satysfing } (2.2)\}} \tag{2.3}$$

Observe that if $M' \subset_a M$ then $K(M') \subset K(M)$.

Claim 1: there is an $M \subseteq \mathbb{N}$ so that

$$K(M') = K(M) \quad \text{for all } M' \subset_a M \tag{2.4}$$

Suppose Claim 1 is false, then using transfinite induction we can construct a family $(M_\alpha)_{\alpha<\omega_1}$ of infinite subsets of N so that for all $\alpha < \beta < \omega_1$ we have $M_\beta \subset_a M_\alpha$ and $K(M_\beta) \subset K(M_\alpha)$. Indeed choose $M_0 = N$; then for successor ordinals, having chosen M_α, simply choose $M_{\alpha+1} \subset_a M_\alpha$ with $K(M_{\alpha+1}) \neq K(M_\alpha)$; for limit ordinals $\gamma < \omega_1$ instead choose M_γ almost contained in M_α for all $\alpha < \gamma$.
Now observe that the corresponding family $(K(M_\alpha))_{\alpha<\omega_1}$ would be a strictly descending sequence of closed set, in contradiction with Remark 2.11. This concludes the proof of Claim 1.
Now choose an M satisfying (2.4). We want to show the following:

Claim 2: for every infinite $M' \subset M$ and every non-empty relatively open subset U of $K(M)$ there exists an infinite $M'' \subseteq M'$ and $y, z \in U$ such that

$$\lim_{m\in M''} f_n(y) \geq r + \delta \quad \text{and} \quad \lim_{m\in M''} f_n(z) \leq r \tag{2.5}$$

Let us fix M' and U as indicated above. By definition of $K(M')$ and (2.4) there is an y in U such that $f_n(y) > r + \delta$ for infinitely many n in M'. Since $(f_n(y))_{n\in M'}$ is a bounded sequence in $\mathbb{R}$, we can choose a subset M_1 of M' such that $(f_n(y))_{n\in M_1}$ converges; again by definition of $K(M_1)$ and (2.4) there is a z in U such that $f_n(y) < r$ for infinitely many n in M_1. To conclude, we simply choose $M'' \subseteq M_1$ such that $(f_n(z))_{n\in M''}$ converges.

To conclude, let $(U_n)_{n\in\mathbb{N}}$ be a base for the topology of $K(M)$. By Claim 2, for each $n \in \mathbb{N}$ we may choose $M_n \subseteq N$, $y_n, z_n \in U_n$ so that 2.5 holds. We may suppose $M_{n+1} \subseteq M_n$ for all $n \in \mathbb{N}$. Now choose Q an infinite subset of $\mathbb{N}$ such that $Q \subset_a M_n$ for all $n \in \mathbb{N}$. Let us define

$$L = \{y_n, z_n \colon n \in \mathbb{N}\}$$

and

$$f(x) = \lim_{n\in Q} f_n(x)$$

for all $x \in L$. Therefore L is a dense subset of $K(M)$; so if U is a non-empty relative subset of L, there is a V relatively open in K such that $V \cap L = U$, hence there is an $i \in \mathbb{N}$ such that $U_i \subseteq V$. Thus $f(y_i) \geq r + \delta$ and $f(z_i) \leq r$ and $y_i, z_i \in U_i \cap L \subseteq U$. Therefore f satisfies the Discontinuity Criterion on L and $(f_{n_k})_{k\in\mathbb{N}} = (f_m)_{m\in Q}$ is the desired subsequence. □

Corollary 2.13. *If $(f_n)_{n\in\mathbb{N}}$ and $(f_{n_k})_{k\in\mathbb{N}}$ are as in Theorem 2.12, then $(f_{n_k})_{k\in\mathbb{N}}$ has no limit points in $\underline{\mathcal{B}}_1(X)$ with the topology of point-wise convergence.*

Proof. If g is any limit point of $(f_{n_k})_{k\in\mathbb{N}}$, then $g = f$ on L, thus satisfies the Discontinuity Criterion. Hence g fails to have a point of continuity in some closed non-empty subset of X, so by Theorem 2.7 $g \notin \underline{\mathcal{B}}_1(X)$. □

Now we can state and prove the main Theorem of Rosenthal's article.

Theorem 2.14. *Let X be a Polish space and let F be a subset of $\underline{\mathcal{B}}_1(X)$. Then the following are equivalent:*

1. *F is relatively compact;*
2. *Every countable infinite subset of F has a limit point in $\underline{\mathcal{B}}_1(X)$;*
3. *Every sequence of elements of F has a convergent subsequence.*

Proof. The implications 1. $\implies$ 2. and 3. $\implies$ 2. are trivial.

2. $\implies$ 3. It is an immediate consequence of Corollary 2.13 and the fact that if 2. holds then F is pointwise bounded.

To conclude, it is sufficient to prove that 2. $\implies$ 1. Suppose 2. holds but 1. does not. Then F is pointwise bounded, hence the pointwise closure of F is a subset of $\prod_{x\in X}[-h_x, h_x]$, where $h_x = \sup\{|f(x)| \colon f \in F\}$. So by Tychonoff Theorem the pointwise closure of F is compact in $X^{\mathbb{R}}$ with the product topology. Therefore, since F is not relatively compact, there must exist a function f in the pointwise closure of F but not Baire-1. By Theorem 2.7, there exists a non-empty closed subset M of X such that $f|_M$ has no point of continuity relative to the topological space M.

Claim: f satisfies the Discontinuity Criterion.
For all $n \in \mathbb{N}$ let us define

$$A_n = \{x \in M \colon \forall U \text{ neighbourhood of } x\ \exists\, y, z \in U \text{ such that } f(y) - f(z) > 1/n\}$$

Since $f|_M$ has no point of continuity it follows $M = \bigcup_{n\in\mathbb{N}} A_n$. Moreover each A_n is closed by definition, so by Baire category Theorem 2.4 there exists $n_0 \in \mathbb{N}$ such that A_{n_0} has non-empty interior, let us call it U_0. Let $\delta = 1/n_0$, then observe that for each non-empty relatively open subset U of M_0, $U \cap U_0$ is a non-empty open neighbourhood of x and so there are $y, z \in U$ such that $f(y) - f(z) > 1/n$. Let $\mathbb{Q} = (r_n)_{n\in\mathbb{N}}$ be an enumeration of the rationals and let

$$B_n = \{x \in M_0 \colon \forall U \text{ neighb. of } x\ \exists\, y, z \in U \cap M_0 \text{ s.t. } f(z) < r_n,\ f(y) > 1/n_0 + r_n\}$$

for all $n \in \mathbb{N}$. From the previous observation it follows that $M_0 = \bigcup_{n\in\mathbb{N}} B_n$. Again all B_n are closed, so by Theorem 2.4 there is $n_1 \in \mathbb{N}$ such that B_{n_1} has non-empty interior, let us call it U_1. Also f satisfies the Discontinuity Criterion on $L = \overline{U_1}$, with $r = r_{n_1}$ and $\delta = 1/n_0$.

Now let $(U_n)_{n\in\mathbb{N}}$ an open basis for the relative topology of L. By the *Claim* for each $n \in \mathbb{N}$ there are $y_n, z_n \in U_n$ such that $f(z) < r$ and $f(y_n) > r + \delta$. Let $Q = \{y_n, z_n \colon n \in \mathbb{N}\}$. Since Q is countable then $\mathbb{R}^Q$ is a metric space, moreover f is in the pointwise closure of F, so there is a sequence $(f_n)_{n\in\mathbb{N}}$ in F such that $\lim_{n\in\mathbb{N}} f_n(q) = f(q)$ for all $q \in Q$. Since Q is dense in L, it follows that $f|_Q$ satisfies the Discontinuity Criterion in Q. To conclude, suppose g is a cluster point of $(f_n)_{n\in\mathbb{N}}$, then $g|_Q = f|_Q$, hence g has non point of continuity on $\overline{Q}$, so by Theorem 2.7 g is not Baire-1. Thus $(f_n)_{n\in\mathbb{N}}$ has no Baire-1 limit points, so 2. fails to hold, a contradiction. □

2.3 A double-dual characterization of separable Banach spaces containing ℓ_1

In this section we will describe the article [OR75], by E. Odell and H. Rosenthal.

Let X be separable Banach space and denote with K the unit ball of X^* endowed with the w^*-topology, then K is a Polish space (see Remark 2.16).

If we see X as canonical embedded in his bidual, from well known facts about the weak* topology, it is known that $X \subseteq C(K)$. Also elements in $X^{**} \setminus X$ are never continuous functions on K.

For this reason one can ask if there are separable Banach spaces where some elements in $X^{**} \setminus X$ could be in $\underline{\mathcal{B}}_1(K)$. Odell and Rosenthal in [OR75] proved that $X^{**} = \underline{\mathcal{B}}_1(K)$ if and only if X contains no subspaces isomorphic to ℓ_1.

Proposition 2.15. *If X is a separable Banach space, then (B_{X^*}, w^*) is metrizable.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a countable dense set in X. Let us define

$$d(x^*, y^*) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x^*(x_n) - y^*(x_n)|$$

It is not difficult to see that d is a metric on (B_{X^*}, w^*) and that the topology induced by d on (B_{X^*}, w^*) is equivalent to the w^*-topology. □

Remark 2.16. Let X be separable Banach space. Throughout this work we will denote by K the unit ball of X^* endowed with the w^*-topology. By Theorem 1.9 K is compact in (X^*, w^*) and by Proposition 2.15 K is metrizable. From this two properties it follows easily that K is also complete and w^*-separable. Thus K is a Polish space.

Definition 2.17. If X is a separable Banach space, we say that $x^{**} \in X^{**}$ is a Baire-1 member of X^{**} is it is the weak*-limit of a sequence of elements of X.

We denote by $\mathcal{B}_1(X)$ the set of all Baire-1 members of X^{**}.

Definition 2.18. Suppose that K is a nonempty compact subset of a locally convex space E, and that μ is a probability measure on K (i.e μ is a non-negative regular Borel measure on K, with $\mu(K) = 1$). A point $x \in E$ is said to be the **barycenter** of μ, if

$$f(x) = \int_K f d\mu$$

for every continuous linear functional f on E.

Theorem 2.19 (Choquet)**.** *Suppose that K is a compact convex subset of a locally convex space E, and that μ is a probability measure on K with barycenter k_0, then*

$$\mu(f) = f(k_0)$$

for each affine function f of first Baire class on K.

The following Lemma is a consequence of Choquet Theorem.

Lemma 2.20. *Let X a Banach space and $K = (B_{X^*}, w^*)$. Then $f \in X^{**}$ is a Baire-1 member of X^{**} if and only if $f|_K$ is a Baire-1 function on K.*

Lemma 2.21. *Let K be a non-empty compact space and f a bounded real-valued function on K having no points of continuity. Then f satisfies the Discontinuity Criterion.*

Proof. For each $n \in \mathbb{N}$, let

$$A_n = \{x \in K : \text{for all } U \text{ neighborhood of } x \; \exists\, y, z \in U \text{ with } f(y) - f(z) > 1/n\}$$

Since f has no points of continuity,

$$K = \bigcup_{n \in \mathbb{N}} A_n$$

By definition A_n is closed for all $n \in \mathbb{N}$, so by the Baire category theorem 2.4 some A_0 has non-empty interior, U_0. Let $K_0 = \overline{U_0}$ and $\delta = 1/no$. We now have that, for any non-empty relatively open subset U of K_0, $U \cap U_0$ is a non-empty open subset of K_0, and hence there exist $y, z \in U$ with $f(y) - f(z) > \delta$. Now let $(r_n)_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$ and define

$$B = \{x \in K_0 \colon \text{for all } U \text{ neighborhood of } x, \text{ there are } y, z \in U \cap K_0 \\ \text{with } f(z) < r_n \text{ and } r + \delta < f(y)\}$$

Since equation 2.1 holds for every non-empty open subset of K_0, $K_0 = \bigcup_{n\in\mathbb{N}} B_n$. Again, all B_n's are closed by definition, so a second application of the category theorem implies there is n_1 such that B_{n_1} has non-empty interior V. Then, putting $L = \overline{V}$ and $r = r_{n_1}$, we have the thesis. $\square$

The following proposition can be deduced putting together some lemmas and a proposition contained in another article by Rosenthal ([Ros74]). The aim of that article was to find a first characterization of Banach spaces containing ℓ_1, and Rosenthal proved that were all and only that containing a bounded sequence with no weak-Cauchy subsequences.

Proposition 2.22. *Let X be a topological space. If $f \colon X \to \mathbb{R}$ satisfies the Discontinuity Criterion and there exists a family $\mathcal{G}$ of continuous and uniformly bounded functions on X such that f is in the pointwise closure of $\mathcal{G}$. Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ that is equivalent to the usual ℓ_1 basis, in the supremum norm.*

Proof. By definition of Discontinuity Criterion, there are $L \subseteq M$ closed and $r, \delta \in \mathbb{R}$ with $\delta > 0$ so that 2.1 holds.

Claim 1: there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ such that, if we define

$$A_n = \{x \in L \colon g_n(x) > r + \delta\} \text{ and } B_n = \{x \in L \colon g_n(x) < r\}$$

then for all F_1, F_2 finite disjoint subsets of $\mathbb{N}$

$$\bigcap_{F_1} A_n \cap \bigcap_{F_2} B_n \neq \emptyset. \tag{2.6}$$

We will use the notation $-A_n = B_n$ so in order to prove equation 2.6 it is sufficient to see that

$$\bigcap_{i=1}^{n} \varepsilon_i A_i \neq \emptyset \quad \text{for all } n \in \mathbb{N} \text{ and for all } \varepsilon_1 \ldots, \varepsilon_n \in \{\pm 1\}.$$

Let's prove the claim by induction. If $n = 1$, f satisfies the Discontinuity Criterion, thus there are $y_1, y_2 \in L$ such that $f(y_1) > r + \delta$ and $f(y_2) < r$. By hypothesis f is in the pointwise closure of $\mathcal{G}$, so there is $g_1 \in \mathcal{G}$ such that $g(y_1) > r + \delta$ and $g(y_2) < r$, hence $\pm A_1 \neq \emptyset$.

Now suppose there are $g_1, \dots, g_{n-1}$ such that $\bigcap_{i=1}^{n-1} \varepsilon_i A_i \neq \emptyset$ for all $\varepsilon_1 \dots, \varepsilon_{n-1} \in \{\pm 1\}$. Let's fix $\overline{\varepsilon} = (\varepsilon_1 \dots, \varepsilon_{n-1}) \in \{\pm 1\}^{n-1}$ then $\bigcap_{i=1}^{n-1} \varepsilon_i A_i$ is a relative open subset of L, so by the Discontinuity Criterion there are $y^{\overline{\varepsilon}}, z^{\overline{\varepsilon}} \in \bigcap_{i=1}^{n-1} \varepsilon_i A_i$ such that $f(y^{\overline{\varepsilon}}) > r+\delta$ and $f(z^{\overline{\varepsilon}}) < r$, so by assumption there is $g_n \in \mathcal{G}$ such that $g_n(y^{\overline{\varepsilon}}) > r+\delta$ and $g_n(z^{\overline{\varepsilon}}) < r$ for all $\overline{\varepsilon} \in \{\pm 1\}^{n-1}$. Thus

$$\bigcap_{i=1}^{n} \varepsilon_1 A_i \cap \varepsilon_n A_n \neq \emptyset$$

for all $\varepsilon_1 \dots, \varepsilon_n \in \{\pm 1\}$ and so Claim 1 is proved.

Claim 2: $(g_n)_{n\in\mathbb{N}}$ is equivalent to the usual ℓ_1 basis.
We need to prove that there exists $c_1, c_2 \in \mathbb{R}^+$ such that for all $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, with $\{n \colon a_n \neq 0\}$ finite, we have

$$c_1 \Big\| \sum_{n\in\mathbb{N}} a_n e_n \Big\|_{\ell_1} \leq \Big\| \sum_{n\in\mathbb{N}} a_n g_n \Big\|_{\infty} \leq c_2 \Big\| \sum_{n\in\mathbb{N}} a_n e_n \Big\|_{\ell_1} \tag{2.7}$$

By assumption $\mathcal{G}$ is uniformly bounded, so there is $c \in \mathbb{R}^+$ such that $\|g_n\|_\infty \leq c$ for all $n \in \mathbb{N}$, hence

$$\Big\| \sum_{n\in\mathbb{N}} a_n g_n \Big\|_{\infty} \leq c \sum_{n\in\mathbb{N}} |a_n| = c \Big\| \sum_{n\in\mathbb{N}} a_n e_n \Big\|_{\ell_1}$$

Therefore we are left to prove the first of the two inequalities in (2.7). In order to do so let us fix a sequence $(c_n)_{n\in\mathbb{N}}$, that WLOG we can assume such that $\sum_{n\in\mathbb{N}} |c_i| = 1$ (otherwise simply put $\overline{c_n} = c_n / \sum_{i\in\mathbb{N}} |c_i|$ for all $n \in \mathbb{N}$ and the following inequalities will remain true). It is sufficient to find an element $s \in L$ such that

$$\frac{\delta}{2} = \frac{\delta}{2} \Big\| \sum_{n\in\mathbb{N}} c_n e_n \Big\|_{\ell_1} \leq \Big| \sum_{n\in\mathbb{N}} c_n g_n(s) \Big| \leq \Big\| \sum_{n\in\mathbb{N}} c_n g_n \Big\|_{\ell_1}. \tag{2.8}$$

Let us consider $C = \{i \in \mathbb{N} \colon c_i > 0\}$ and $D = \{i \in \mathbb{N} \colon c_i < 0\}$, which are two finite disjoint subsets of $\mathbb{N}$, so by (2.6) there exists an element $x \in \bigcap_{n\in C} A_n \cap \bigcap_{n\in D} B_n$ and an element $y \in \bigcap_{n\in D} A_n \cap \bigcap_{n\in C} B_n$. Suppose $r > 0$ (the case $r < 0$ is analogous), and consider $D' = \{i \in B \colon g_i(x) > 0\}$, then

$$\sum_{n\in D} c_n g_n(x) \geq \sum_{n\in D'} c_n g_n(x) \geq -r \sum_{n\in D'} |c_n| \geq -r \sum_{n\in D} |c_n|$$

and similarly

$$-\sum_{n\in C} c_n g_n(y) \geq -r \sum_{n\in C} |c_n|$$

Hence

$$\sum_{n\in\mathbb{N}} c_n g_n(x) = \sum_{n\in C} c_n g_n(x) + \sum_{n\in D} c_n g_n(x) > (r+\delta) \sum_{n\in C} |c_n| - r \sum_{n\in D} |c_n| \tag{2.9}$$

and

$$-\sum_{n\in\mathbb{N}} c_n g_n(y) = -\sum_{n\in C} c_n g_n(y) - \sum_{n\in D} c_n g_n(y) > -r\sum_{n\in C}|c_n| + (r+\delta)\sum_{n\in D}|c_n| \quad (2.10)$$

Therefore, adding up (2.9) and (2.10) we obtain

$$\sum_{n\in\mathbb{N}} c_n g_n(x) - \sum_{n\in\mathbb{N}} c_n g_n(y) > \delta \sum_{n\in\mathbb{N}} |c_n| = \delta$$

thus $\sum_{n\in\mathbb{N}} c_n g_n(x) > \delta/2$ or $-\sum_{n\in\mathbb{N}} c_n g_n(y) > \delta/2$ so (2.8) holds chosing $s = x$ or $s = y$. □

We are now ready to prove the main result of [OR75].

Theorem 2.23. *A separable Banach space X contains a subspace isomorphic to ℓ_1 if and only if there exists an element $x^{**} \in X^{**}$ such that there is no sequence $(x_n)_{n\in\mathbb{N}}$ in X that weakly* converges to x^{**}.*

Proof. The "only if" assertion is immediate form Remark 1.6.

Now assume that there exists an element $x^{**} \in X^{**}$ such that there is no sequence $(x_n)_{n\in\mathbb{N}}$ in X that weakly* converges to x^{**}, i.e. x^{**} is not a Baire-1 member of X^{**}. Let $K = (B_{X^*}, w^*)$. By Lemma 2.20 $x^{**}|_K$ is not in $B_1(K)$, hence by Baire characterization theorem 2.7 there is a closed subspace M of K such that $x^{**}|_M$ has no point of continuity. Thus by Lemma 2.21 f satisfies the Discontinuity Criterion on M, i.e there are $L \subseteq M$ closed and $r, \delta \in \mathbb{R}$ with $\delta > 0$ so that 2.1 holds. WLOG we can suppose $\|x^{**}\| = 1$, so by Goldstine Theorem 1.10 there is a net in B_X weak* convergent to x^{**}. Consequently, if we define

$$\mathcal{G} = \{g \in C(L) \colon \text{ there is } x \in B_X \text{ such that } g = x|_L\}$$

clearly $x^{**}|_L$ is in the pointwise closure of $\mathcal{G}$. Hence, by Proposition 2.22 there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ equivalent to the usual ℓ_1 basis. Now, by definition of $\mathcal{G}$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that $g_n = x_n|_L$ for all $n \in \mathbb{N}$, then the subspace $\overline{\operatorname{span}}\{x_n \colon n \in \mathbb{N}\}$ of X is isomorphic to ℓ_1. □

The previous Theorem can be put in a broader characterization.

Theorem 2.24 (Rosenthal-Odell)**.** *Let X be a separable Banach space. Then the following assertions are equivalent:*

1. *X contains an isomorphic copy of ℓ_1;*
2. *There is $x^{**} \in X^{**}$ which is not weak* limit of any sequence in X;*
3. *There is a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X with no weak-Cauchy subsequences;*

4. *There is a bounded sequence* $(x_n^{**})_{n\in\mathbb{N}}$ *in* X^{**} *with no weak***-Cauchy subsequences.*

Proof. 1. $\iff$ 2. is Theorem 2.23

4. $\implies$ 3. and 1 $\implies$ 4 are trivial.

3. $\implies$ 1. By assumption there is a sequence $(g_n)_{n\in\mathbb{N}} \subseteq X^{**}$ with $\|g_n\|_{X^{**}} \leq 1$ and with no weak*-Cauchy subsequences. Define

$$\mathcal{F} = \{x^{**}|_K : x^{**} \in X^{**},\ \|x^{**}\|_{X^{**}} \leq 1\}$$

where $f_n = g_n|_K \in \mathcal{F}$ for all $n \in \mathbb{N}$, has no pointwise convergent subsequences. Hence, by Theorem 2.12 there exists a subset L of K and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ pointwise convergent on L towards a function f which satisfies the Discontinuity Criterion. By Goldstine Theorem 1.10 there is a sequence $(h_{n_k})_{k\in\mathbb{N}} = (x_{n_k}|_K)_{k\in\mathbb{N}}$ with $x_{n_k} \in B_X$ for all $k \in \mathbb{N}$ pointwise convergent to f. Let

$$\mathcal{G} = \{g \in C(L) : \text{ there is } x \in B_X \text{ such that } g = x|_L\}$$

therefore f is in the pointwise closure of $\mathcal{G}$ and we can conclude applying Propostion 2.22. □

Chapter 3

Bourgain Theorem on Lavrentiev index

In [Bou80a] Bourgain worked on an interesting result about the Lavrentiev index. The main Theorem of this work is "an improvement of Theorem 2.24, relating the topological nature of double-dual elements to the ℓ_1-ordinals of the space" as himself writes in his article.

First we need an introduction on trees.

3.1 Trees

The concept of a tree is a basic combinatorial tool in descriptive set theory. What is referred to as a tree in this subject is not, however, the same notion as the one used either in graph theory.

Definition 3.1. Let X be an arbitrary set. A **tree** T on X is a subset of $\bigcup_{n\in\mathbb{N}} X^n$ such that if $(x_1, \ldots, x_{n+1}) \in T$ then $(x_1, \ldots, x_n) \in T$.

Definition 3.2. A tree T on X is called **well-founded** if there is no sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(x_1, \ldots, x_n) \in T$ for all $n \in \mathbb{N}$.
A tree T on X is called **ill-founded** if it is not well-founded.

We will now describe the process of *pruning* a tree using transfinite induction. Let's define $T^0 = T$. For successor ordinals, suppose T^α is already defined and let

$$T^{\alpha+1} = \bigcup_{n\in\mathbb{N}} \{(x_1, \ldots, x_n) \in X^n \colon \text{there is an } x \in X \text{ such that } (x_1, \ldots, x_n, x) \in T\}$$

For limit ordinals define instead

$$T^\gamma = \bigcap_{\alpha<\gamma} T^\alpha$$

Notice that this way the T^α are strictly decreasing, hence for α large enough, T^α will be empty.

Definition 3.3. If T is a well-founded tree on X, the **order** of T, is the smallest ordinal $o[T]$ such that $T^{o[T]} = \emptyset$. If T is ill-founded we say that $o[T] = \omega_1$.

Theorem 3.4 (Kuren-Martin)**.** *Let T a well-founded tree on a Polish space X, then $o[T] < \omega_1$.*

If T is a tree on X, we will denote by T_n the subset $T \cap X^n$, i.e. the set of all sequence of length n in T.

Definition 3.5. Let S be a tree on X and T be a tree on Y. A map $\rho\colon S \to T$ is called **regular** if it preserves the length and the order, i.e.

- $\rho(S_n) \subseteq T_n$
- if $\rho(x_1, \dots, x_n, x_{n+1}) = (y_1, \dots, y_n, y_{n+1})$ then $\rho(x_1, \dots, x_n) = (y_1, \dots, y_n)$

Proposition 3.6. *If T and S are two well-founded trees and if there is a regular map $\rho\colon S \to T$ then $o[S] \leq o[T]$.*

Proposition 3.7. *Let $T \in \mathcal{WF}$ with $o[T] > 1$. Then*

$$o[T] = \sup_{k \in \omega} o[T(k)] + 1.$$

Proof. It is an easy exercise by transfinite induction. $\square$

3.2 Lavrentiev index

In this section, K will denote a compact metric space and A, B will be two disjoint G_δ subset of K. For more details and motivation on Lavrentiev index we refer the reader the article [M25].

Definition 3.8. Let K and A, B as above. We will denote by $R(A, B)$ the family of all strictly increasing net $(G_\alpha)_{\alpha \leq \beta}$ $(\beta < \omega_1)$ of open subset of K such that:

- $G_0 = \emptyset$ and $G_\beta = K$;
- $G_\gamma = \bigcup_{\alpha < \gamma} G_\alpha$ if γ is a limit ordinal;
- $G_{\alpha+1} \setminus G_\alpha$ is disjoint with either A or B for all $\alpha < \beta$.

Notice that $R(A, B)$ is always non-empty. For example see [KK68].

Definition 3.9. We call the **Lavrentiev index** on A, B the ordinal number

$$L(A,B) = \min\{\beta < \omega_1 \colon \text{ there is } (G_\alpha)_{\alpha \leq \beta} \in R(A,B)\}$$

If f is a function in $\mathcal{B}_1(K)$ and $a, b \in \mathbb{R}$ with $a < b$ then $A = \{f \leq a\}$ and $B = \{f \geq b\}$ are two disjoint G_δ. Indeed, if $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C(K)$ pointwise convergent to f, then we have

$$X \setminus A = \bigcup_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \left\{ x \in K \colon f_m(x) \geq a + \frac{1}{k} \right\}.$$

Thus $X \setminus A$ is an F_σ set and therefore A is G_δ. For B the proof is analogous. Therefore we can give the following definition.

Definition 3.10.

$$L(f,a,b) = L(\{f \leq a\}, \{f \geq b\})$$

3.3 Bourgain Theorem on Lavrentiev index

Proposition 3.11. *Let A and B be disjoint G_δ subsets of K and let $(O_n)_n$ be a sequence of open sets in K so that either $A \subset \bigcup_{n \in \mathbb{N}} O_n$ or $B \subset \bigcup_{n \in \mathbb{N}} O_n$. Then*

$$\begin{aligned} L(A,B) &\leq L\left(A \cap O_1, B \cap O_1\right) + \ldots + L\left(A \cap O_n, B \cap O_n\right) + \ldots + 1 \\ &\leq \sup_n L\left(A \cap O_n, B \cap O_n\right) \cdot \omega_0 + 1 \end{aligned}$$

Proof. Let $\beta_n = L\left(A \cap O_n, B_n \cap O_n\right)$ and let $(G^n_\alpha)_{\alpha \leq \beta_n}$ be a member of $R\left(A \cap O_n\right), B \cap O_n)$. Take $\beta = \beta_1 + \ldots + \beta_n + \ldots$, which is a limit ordinal and consider the increasing sequence $(G_\alpha)_{\alpha \leq \beta + 1}$ of open sets in K, obtained as following:

- $\mathrm{G}_\alpha = O_1 \cap \mathrm{G}^1_\alpha$ if $\alpha \leqslant \beta_1$;
- $\mathrm{G}_{\beta_1 + \ldots + \beta_n + 1 + \alpha} = O_1 \cup \ldots \cup O_n \cup (O_{n+1} \cap G^{n+1}_\alpha)$ if $n \in \mathbb{N}$ and $\alpha < \beta_{n+1}$;
- $G_\beta = \bigcup_{n \in \mathbb{N}} O_n$ and $G_{\beta+1} = K$.

By hypothesis $\mathrm{G}_{\beta+1} \backslash \mathrm{G}_\beta$ is either disjoint with A or with B. It is easily seen that $(G_\alpha)_{\alpha \leq \beta+1}$ verifies the properties of Definition 3.8 and hence belongs to $R(A,B)$. Thus $L(A,B) \leq \beta + 1$. □

Proposition 3.12. *Assume A and B disjoint G_δ subsets of K and $(A_n)_n$ and $(B_n)_n$ be two sequences of open sets satisfying $A \subset \bigcup_m \bigcap_{n \geq m} A_n$ and $B \subset \bigcup_m \bigcap_{n \geq m} B_n$. Let $X_1, \ldots, X_d$ be a finite family of open subsets of K and $\alpha < \omega_1$, such that*

$$L\left(A \cap X_1, B \cap X_1\right) > \alpha_1 \cdot \omega_0 + 1$$

for all $i = 1, \ldots, d$.

Then there exists some $n \in \mathbb{N}$ so that $L\left(A \cap X_i \cap \varepsilon A_n, B \cap X_i \cap \varepsilon A_n\right) > \alpha$ whenever $i = 1, \ldots, d$ and $\varepsilon = \pm 1$.

Proof. Suppose that the statement is untrue, then for each $n \in \mathbb{N}$ there is some $i_n = 1, \ldots, \mathrm{d}$ and $\varepsilon_n = \pm 1$ such that

$$L\left(A \cap X_{i_n} \cap \varepsilon_n A_n, B \cap X_{i_n} \cap \varepsilon_n A_n\right) \leq \alpha$$

Clearly there is some $i = 1, \ldots, \mathrm{d}$ and some $\varepsilon = \pm 1$ so that $N = \{n \in \mathbb{N}; i_n = i, \varepsilon_n = \varepsilon\}$ is an infinite set. Since $(\varepsilon A_n)_{n \in N}$ is an open covering of either A or B, Proposition 3.11 yields that $L\left(A \cap X_i, B_1 \cap X_i\right) \leqslant \alpha \cdot \omega_0 + 1$.

This is the required contradiction. $\square$

Let us introduce the following notation. Define by transfinite induction the ordinal $[\alpha]$:

- $[0] = 0$;
- $[\beta] = \sup_{\alpha<\beta}[\alpha] \cdot \omega_0 + 1$ for all $\beta < \omega_1$.

Definition 3.13. Let $\alpha < \omega_1$ be an ordinal number. We define

$$T(\alpha) = \bigcup_{n \in \mathbb{N}} \{(\alpha_1, \ldots, \alpha_m) \colon \alpha > \alpha_1 > \alpha_2 > \ldots > \alpha_n\}$$

Observe that $o[T(\alpha)] = \alpha$ for all ordinal numbers $\alpha < \omega_1$

Let us denote by $\omega_0^{<\omega_0}$ the tree on $\mathbb{N}$ consisting of all finite increasing sequences on $\mathbb{N}$, i.e.

$$\omega_0^{<\omega_0} = \bigcup_{m \in \mathbb{N}} \{(n_1, \ldots, n_m) \in \mathbb{N}^m \colon n_1 < n_2 < \ldots < n_m\}$$

.

Proposition 3.14. *Let A and B be two disjoint G_δ sets and $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ be two sequences of open subset of K satisfying*

$$A \subset \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} A_n \quad \text{and} \quad B \subset \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} B_n$$

Let $\alpha < \omega_1$ be such that $L(A, B) > [\alpha]$. Then there is a regular map

$$\rho \colon T(\alpha) \to \omega_0^{<\omega_0}$$

such that

$$\rho(\alpha_1, \ldots, \alpha_k) = (n_1, \ldots, n_k) \Rightarrow L(A \cap \bigcap_{l=1}^{k} \varepsilon_l A_{n_l}, B \cap \bigcap_{l=1}^{k} \varepsilon_l A_{n_l}) > [\alpha_k] \tag{3.1}$$

for all $(\varepsilon_1, \ldots, \varepsilon_k) \in \{-1, +1\}^k$.

Proof. We define ρ on $T_k(\alpha)$ by induction on k.

Fix α_1 in $T_1(\alpha)$. Since $L(A,B) > \alpha_1 \cdot \omega_0 + 1$, by Proposition 3.12 there is some $n_1 \in \mathbb{N}$ such that $L\left(A \cap \epsilon_1 A_{n_1}, B \cap \varepsilon_1 A_{n_1}\right) > [a_1]$ for $\varepsilon_1 = \pm 1$. Hence, take $\rho\left(\alpha_1\right) = n_1$.

Assume now ρ defined on $T_k(\alpha)$ and fix $(\alpha_1, \dots, \alpha_k, \alpha_{k+1}) \in \mathcal{T}_{k+1}(\alpha)$. Then $(\alpha_1, \dots, \alpha_k) \in T_k(\alpha)$ and the induction hypothesis applies. Hence, if $\rho\left(\alpha_1, \dots, \alpha_k\right) = (n_1, \dots, n_k)$, then since $[\alpha_k] \geq [\alpha_{k+1}] \cdot \omega_0 + 1$, by Proposition 3.12 again there is some $n_{k+1} > n_k$ so that

$$L\left(A \cap \bigcap_{l=1}^{k+1} \varepsilon_l A_{n_l}, B \cap \bigcap_{l=1}^{k+1} \varepsilon_l A_{n_l}\right) > [\alpha_{k+1}]$$

for $(\varepsilon_1, \dots, \varepsilon_k, \varepsilon_{k+1}) \in \{1, -1\}^{k+1}$.

We only have to take

$$\rho\left(\alpha_1, \dots, \alpha_k, \alpha_{k+1}\right) = (n_1, \dots, n_k, n_{k+1})$$

to complete the construction.

It is easily seen that ρ is a regular map that satisfies the required condition. $\square$

Definition 3.15. Let $(A_n, B_n)_{n \in \mathbb{N}}$ be a sequence of pairs of subset of K. Using the notation $-A_n = B_n$ let us define for all $n \in \mathbb{N}$ the tree

$$T(A_n, B_n, n) = \bigcup_{k \in \mathbb{N}} \Bigg\{ (n_1, \dots, n_k) \in \mathbb{N}^k \colon n_1 < n_2 < \cdots < n_k \text{ and } \bigcap_{l=1}^{k} \varepsilon_l A_{n_l} \neq \emptyset \text{ for all } \varepsilon_1, \dots, \varepsilon_n \in \{\pm 1\}^n \Bigg\}$$

Define also $o(A_n, B_n, n) = o[T(A_n, B_n, n)] + 1$

Proposition 3.16. *Let A and B be two disjoint G_δ sets and $(A_n)_{n \in \mathbb{N}}$, $(B_n)_{n \in \mathbb{N}}$ be two sequences of open subset of K satisfying*

$$A \subset \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} A_n \quad \text{and} \quad B \subset \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} B_n$$

Then $L(A,B) \leq [o(A_n, B_n, n)]$.

Proof. Take $\alpha = o(A_n, B_n, n)$ and suppose $L(A,B) > [\alpha]$. Consider $\rho \colon T(\alpha) \to \omega_0^{<\omega_0}$ a regular map satisfyng (3.1). Since $\rho(T(\alpha)) \subset T(A_n, B_n, n)$, it follows that $o(A_n, B_n, n) \geq o[T(\alpha)] + 1 = \alpha + 1$, which is a contradiction. $\square$

Theorem 3.17. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $C(K)$ pointwise convergent to f and take real numbers $a < b < c < d$. Then*

$$L(f, a, b) \leq [o(\{f_n < c\}, \{f_n > d\}, n)]$$

Proof. It follows from the fact that

$$\{f \le a\} \subset \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \{f_n < c\} \quad \text{and} \quad \{f \le b\} \subset \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \{f_n > d\}$$

and by Proposition 3.16. □

Corollary 3.18. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C(K)$ which is pointwise relatively compact in $B_1(K)$ and let f be a cluster point of $(f_n)_{n\in\mathbb{N}}$ with the pointwise topology of $B_1(K)$. Take real numbers $a < b < c < d$. Then*

$$L(f, a, b) \le [o(\{f_n < c\}, \{f_n > d\}, n)] < \omega_1$$

Proof. By Rosenthal Theorem 2.12 there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ point-wise convergent to f. Thus by Theorem 3.17

$$L(f, a, b) \le [o(\{f_{n_k} < c\}, \{f_{n_k} > d\}, n_k)] \le [o(\{f_n < c\}, \{f_n > d\}, n)] < \omega_1$$

□

Definition 3.19. Let X be a separable Banach space. For all $\delta > 0$ let us define

$$T(X, \delta) = \bigcup_{n=1}^{\infty} \left\{ (x_1, \dots, x_n) \in X^n \colon \|x\| \le 1 \text{ and } \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|_X \ge \delta \sum_{i=1}^{n} |\lambda_i| \right\}$$

Observe that all such $T(x, \delta)$ are closed trees of X. Moreover if $\ell_1 \subseteq X$ than there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\delta \left\| \sum_{i=1}^{n} \lambda_i e_i \right\|_{\ell_1} = \delta \sum_{i=1}^{n} |\lambda_i| \le \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|_X \le \sum_{i=1}^{n} |\lambda_i| = \left\| \sum_{i=1}^{n} \lambda_i e_i \right\|_{\ell_1}$$

for all $n \in \mathbb{N}$ and for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, hence $(x_1, \dots, x_n)$ is in $T(x, \delta)$ for all $n \in \mathbb{N}$, i.e. $T(x, \delta)$ is ill-founded. Similarly, if there is a $\delta > 0$ such that $T(x, \delta)$ is ill-founded, then his infinite branch is a sequence $(x_n)_{n\in\mathbb{N}}$ in X equivalent to the usual ℓ_1 basis. Everything said before can be summarised in the following Proposition.

Proposition 3.20. *$T(x, \delta)$ is well-founded if and only if X does not contain subspaces isomorphic to ℓ_1.*

Definition 3.21. Let X be a separable Banach space such that $\ell_1 \not\subseteq X$ and let $\delta > 0$. We will call the **Bourgain index** of X and δ,

$$O(X, \delta) = o[T(X, \delta)] + 1$$

Notice that $O(X, \delta) < \omega_1$, since $T(X, \delta)$ is well-founded by Proposition 3.20.

Lemma 3.22. *Let $x_1, \dots, x_n \in X$ and $a, b \in \mathbb{R}$ with $a < b$. For all $l \in \{1, \dots, k\}$ define*

$$A_l = \{x^* \in K \colon x^*(x_l) < a\} \quad \textit{and} \quad B_l = \{x^* \in K \colon x^*(x_l) > b\}$$

If $\bigcap_{l=1}^k \varepsilon_l A_l \neq \emptyset$ for all $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$ (we are using the notation $B_l = -A_l$), then

$$\left\| \sum_{l=1}^k \lambda_i x_i \right\|_X \geq \frac{b-a}{2} \sum_{l=1}^k |\lambda_l| \tag{3.2}$$

for all $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$.

Proof. Fix $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ and put

$$\begin{cases} \varepsilon_l = +1, \delta_l = -1 & \text{if } \lambda_l \geq 0 \\ \varepsilon_l = -1, \delta_l = +1 & \text{if } \lambda_l < 0 \end{cases}$$

By assumption there exist

$$x^* \in \bigcap_{l=1}^k \varepsilon_l A_l \quad \text{and} \quad y^* \in \bigcap_{l=1}^k \delta_l A_l$$

Therefore

$$\begin{aligned} \left\| \sum_{l=1}^k \lambda_l x_l \right\|_{\ell_1} \geq \left| x^*(\sum_{l=1}^k \lambda_l x_l) \right| &\geq -\sum_{\lambda_l \geq 0} |\lambda_l| x^*(x_l) + \sum_{\lambda_l < 0} |\lambda_l| x^*(x_l) \\ &\geq -\sum_{\lambda_l \geq 0} |\lambda_l| a + \sum_{\lambda_l < 0} |\lambda_l| b \end{aligned} \tag{3.3}$$

Similarly, using y^* we also have:

$$\left\| \sum_{l=1}^k \lambda_l x_l \right\|_{\ell_1} \geq + \sum_{\lambda_l \geq 0} |\lambda_l| b - \sum_{\lambda_l < 0} |\lambda_l| a \tag{3.4}$$

Adding up (3.3) and (3.4) we obtain the thesis. □

Theorem 3.23 (Bourgain)**.** *Let X be a separable Banach space which does not contain a copy of ℓ_1. If $x^{**} \in X^{**}$ with $\|x^{**}\| \leq 1$, $a, b \in \mathbb{R}$ and $0 < \delta < \frac{b-a}{2}$, then $L(x^{**}|_K, a, b) \leq [o(X, \delta)]$ holds.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a dense sequence in B_X. By Theorem 2.23 $(x_n|_K)_{n \in \mathbb{N}}$ is relatively compact in $B_1(K)$. Let $c, d \in \mathbb{R}$ such that $a < b < c < d$ and $\delta < \frac{d-c}{2}$. Let us define

$$A_n = \{x^* \in K \colon x^*(x_n) < c\} \quad \text{and} \quad B_n = \{x^* \in K \colon x^*(x_n) > d\}$$

Then by Lemma 3.22 the map $n \mapsto x_n$ induces the regular map $\rho\colon T(A_n, B_n, n) \mapsto o[T(X, \delta)]$. Hence by Proposition 3.6

$$o[T(A_n, B_n, n)] \leq T(X, \delta)$$

thus

$$o(A_n, B_n, n) \leq o(X, \delta).$$

To conclude, by Corollary 3.18

$$L(x^{**}|_K, a, b) \leq [o(A_n, B_n, n)] \leq [o(X, \delta)]$$

□

Chapter 4

Bourgain example

This chapter is based on some unpublished notes by Bourgain, which continue his work [Bou80b]. In these notes he further investigated on the topic, answering some question posed by Rosenthal on two subclasses of first Baire functions introduced by Bourgain.

4.1 Results on the bidual

The following Lemma is a consequence of Choquet Theorem 2.19.

Lemma 4.1. *Let X be a separable Banach space, let $x^{**} \in \underline{\mathcal{B}}_1(K)$ and define*

$$\begin{aligned} i : X &\longrightarrow C(K) \\ x &\longmapsto x|_K \end{aligned} \tag{4.1}$$

If $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ is pointwise convergent to x^{**} then $(f_n)_{n\in\mathbb{N}}$, seen as a sequence in $C(K)^{**}$, weakly* converges to $i^{**}(x^{**})$.

Lemma 4.2. *Let A and B two subsets of a Banach space X. If $\overline{A}^{w^*} \cap \overline{B}^{w^*} \neq \emptyset$ then $d(\mathrm{co}(A), \mathrm{co}(B)) = 0$.*

The following is a strict improvement of Odell-Rosenthal Theorem 2.23.

Theorem 4.3. *Let X be a Banach space, $K = (B_{X^*}, w^*)$ and $x^{**} \in X^{**}$. Then the following conditions are equivalent:*

1.) $x^{**} \in \underline{\mathcal{B}}_1(K)$.

2.) There is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that $x^{**} = w^*\text{-}\lim x_n$ and $\|x_n\|_{X^{**}} \leq \|x^{**}\|_{X^{**}}$.

Proof. Since X is a subset of $C(K)$, it is trivial that 2.) implies 1.).

For the converse, suppose $x^{**} \in \mathcal{B}_1(K)$ i.e. there is $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ pointwise convergent to x^{**}. Let's define $\mathcal{C} = B_X(0_X, \|x^{**}\|_{X^{**}})$ and $\mathcal{D}_m = \mathrm{co}\{f_n : n \geq m\}$. By Lemma 4.1

$$i^{**}(x^{**}) \in \overline{\mathcal{D}_n}^{w^*}$$

for all $n \in \mathbb{N}$. By Goldstine's Theorem 1.10, $i(\mathcal{C})$ is w^*-dense in $B(0_{C(K)^{**}}, \|x^{**}\|_{X^{**}})$, thus

$$i^{**}(x^{**}) \in \overline{i(\mathcal{C})}^{w^*}$$

Since $\overline{\mathcal{D}_n}^{w^*} \cap \overline{i(\mathcal{C})}^{w^*} \neq \emptyset$, by Lemma 4.2 $d(\mathcal{D}_n, i(\mathcal{C})) = 0$ for all $n \in \mathbb{N}$, i.e. there is a $x_n \in \mathcal{C}$ such that $d(i(x_n), \mathcal{D}_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ built in this way is pointwise convergent to x^{**} and $\|x_n\|_{X^{**}} \leq \|x^{**}\|_{X^{**}}$. □

Lemma 4.4. *Let X be a Banach space, $(x_n)_{n\in\mathbb{N}} \subseteq X$ such that w^*-$\lim x_n = x^{**} \in X^{**}$ and $d(x^{**}, X) > \varepsilon > 0$. Then exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that*

$$\|a_1 x_{n_1} + a_2(x_{n_2} - x_{n_1}) + \ldots + a_m(x_{n_m} - x_{n_{m-1}})\| \geq \frac{\varepsilon}{2} \max_{1\leq i\leq m} |a_i| \tag{4.2}$$

for all $(a_1, \ldots, a_m) \in \mathbb{R}^m$.

Corollary 4.5. *If the hypothesis of Lemma 4.4 holds then*

$$\|a_1 x_{n_1} + a_2 x_{n_2} + \ldots + a_m x_{n_m}\| \geq \frac{\varepsilon}{4} \max_{1\leq i\leq m} |a_i| \tag{4.3}$$

for all $(a_1, \ldots, a_m) \in \mathbb{R}^m$.

Proof. It is sufficient to define $b_k = a_k + \ldots + a_m$ for all $k \in \{1, \ldots, m\}$ and to rewrite (4.2) using the n-tuple $(b_1, \ldots, b_k)$. □

Lemma 4.6. *Let E and F be two closed subspaces of a Banach space X, and let $i\colon E \hookrightarrow X$ and $j\colon F \hookrightarrow X$ be the canonical immersions. Suppose there is an $x^{**} \in X^{**}$ such that x^{**} is in $\mathcal{B}_1(K)$ and there exist $e^{**} \in E^{**}, f^{**} \in F^{**}$ such that $i^{**}(e^{**}) = j^{**}(f^{**}) = x^{**}$. Then there are a closed subspace Z of E which is linear isomorphic to a subspace of F and a sequence $(z_n)_{n\in\mathbb{N}} \subseteq Z$ w^*-convergent to x^{**} in Z^{**} such that $\|z_n\|_{X^{**}} \leq \|x^{**}\|_{X^{**}}$.*

Proof. Suppose $x^{**} \in X^{**} \setminus X$, otherwise the thesis is trivial. By Lemma 4.1 there are $(e_n)_{n\in\mathbb{N}} \subseteq E$ and $(f_n)_{n\in\mathbb{N}} \subseteq F$ weakly* convergent to x^{**}. Define

$$\mathcal{C}_m = \mathrm{co}\{e_n : n \geq m\} \qquad \text{and} \qquad \mathcal{D}_m = \mathrm{co}\{f_n : n \geq m\}$$

for all $m \in \mathbb{N}$.

Of course $x^{**} \in \overline{\mathcal{C}_m}^{w^*} \cap \overline{\mathcal{D}_m}^{w^*}$ for all $m \in \mathbb{N}$, so by Lemma 4.2 $d(\mathcal{C}_n, \mathcal{D}_n) = 0$. Hence there are $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}_n$ and $(v_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_n$ such that

$$\|u_n - v_n\|_X < \frac{\varepsilon}{2^{n-3}} \tag{4.4}$$

Since $d(x^{**}, X) > \varepsilon > 0$, Corollary 4.5 can be applied, so there is a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$\max_{1\leq i\leq m} |a_i| \leq \frac{4}{\varepsilon}\left\| \sum_{i=1}^{l} a_i u_{n_i} \right\| \tag{4.5}$$

Let $Z = \overline{\operatorname{span}}\{u_{n_k} \colon k \in \mathbb{N}\}$. By (4.4), Z is isomorphic to $\overline{\operatorname{span}}\{v_{n_k} \colon k \in \mathbb{N}\}$. To conclude, it is clear that $(u_{n_k})_{k\in\mathbb{N}} \subseteq Z$ weakly* converges to x^{**}, thus by Theorem 4.3 there is a sequence $(z_n)_{n\in\mathbb{N}}$ with the wanted properties. □

4.2 Two sub-classes of first Baire functions

If X is a separable Banach space, let us recall that $\mathcal{B}_1(X)$ is the set of all elements in X^{**} that are weak*-limit of a sequence in X. By Theorem 4.3, we can write

$$\mathcal{B}_1(X) = \{x^{**} \in X^{**} \colon \exists (x_n)_{n\in\mathbb{N}} \subseteq X \text{ s.t. } \|x_n\|_{X^{**}} \leq \|x^{**}\|_{X^{**}},\ w^*\text{-}\lim x_n = x^{**}\}$$

Now, let us define two subsets of $\mathcal{B}_1(X)$.

Definition 4.7.

$$\begin{aligned}\mathcal{B}_1^0(X) = \{x^{**} \in X^{**} \colon\ & \text{there exists } (x_n)_{n\in\mathbb{N}} \subseteq X \text{ such that } \|x_n\|_{X^{**}} \leq \|x^{**}\|_{X^{**}},\\ & x^{**} = w^*\text{-}\lim x_n \text{ and } \overline{\operatorname{span}}\{x_n \colon n \in \mathbb{N}\} \text{ has separable dual}\};\end{aligned}$$

$$\begin{aligned}\mathcal{B}_1^1(X) = \{x^{**} \in X^{**} \colon\ & \text{there exists } (x_n)_{n\in\mathbb{N}} \subseteq X \text{ such that } \|x_n\|_{X^{**}} \leq \|x^{**}\|_{X^{**}},\\ & x^{**} = w^*\text{-}\lim x_n \text{ and } \ell_1 \not\hookrightarrow \overline{\operatorname{span}}\{x_n \colon n \in \mathbb{N}\}\}.\end{aligned}$$

Theorem 4.8. *If X is a Banach space and $\ell_1 \hookrightarrow X$ then X^* is not separable.*

Corollary 4.9. *If X is a separable Banach space then $\mathcal{B}_1^0(X) \subseteq \mathcal{B}_1^1(X)$.*

Thus the following chain of inclusions holds:

$$X \subseteq \mathcal{B}_1^0(X) \subseteq \mathcal{B}_1^1(X) \subseteq \mathcal{B}_1(X) \subseteq X^{**}.$$

Remark that by Theorem 2.24, if X contains no isomorphic copies of ℓ_1, then $\mathcal{B}_1^1(X) = \mathcal{B}_1(X) = X^{**}$. Moreover, by Remark 1.6, $\ell_1 = \mathcal{B}_1^0(\ell_1) = \mathcal{B}_1^1(\ell_1) = \mathcal{B}_1(\ell_1)$.

Thus the following question arises:

Question 4.10. Does the equality $\mathcal{B}_1(X) = \mathcal{B}_1^1(X)$ hold for every separable Banach space?

Proposition 4.11. *Let X be a separable Banach space, let $x^{**} \in X^{**}$. Then*

I) If there is $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ pointwise convergent to x^{**} and such that $\overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$ has not separable dual, then $x^{**} \in \mathcal{B}_1^0(K)$;

II) If there is $(f_n)_{n\in\mathbb{N}} \subseteq C(K)$ pointwise convergent to x^{**} and such that $\overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$ does not contain ℓ_1, then $x^{**} \in \mathcal{B}_1^1(K)$.

Proof. Let $Y = \overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$, let $i: X \to C(K)$ and $j: Y \to C(K)$ be the immersions as in (4.1). By Lemma 4.1 $(f_n)_{n\in\mathbb{N}} \subseteq C(K)^{**}$ is weakly* convergent to $i^{**}(x^{**})$, so by Theorem 4.3 $i^{**}(x^{**}) \in \mathcal{B}_1(K)$. On the other hand $i^{**}(x^{**}) \in j^{**}(Y^{**})$. Hence by Lemma 4.6 there is a closed subspace Z of X linear isomorphic to a subspace of Y and a sequence $(z_n)_{n\in\mathbb{N}} \subseteq Z$ w*-convergent to x^{**} in Z^{**} such that $\|z_n\|_{X^{**}} \leq \|x^{**}\|_{X^{**}}$.

Therefore if I) holds, then Z^* is separable, thus $x^{**} \in \mathcal{B}_1^0(K)$.
If II) holds, then $\ell_1 \not\hookrightarrow Z$, thus $x^{**} \in \mathcal{B}_1^0(K)$. □

4.3 Bourgain example

We will now give an example of X for which $X \subsetneq \mathcal{B}_1^1(X) \subsetneq \mathcal{B}_1(X)$ and also $X \subsetneq \mathcal{B}_0^1(X) \subsetneq \mathcal{B}_1(X)$

First we need this Proposition, whose proof is omitted.

Proposition 4.12. *Let K be a compact metric space and $(f_n)_{n\in\mathbb{N}}$ in $C(K)$ pointwise stabilized. Then $\overline{\operatorname{span}}\{f_n : n \in \mathbb{N}\}$ has separable dual.*

Proposition 4.13. *Let K be a compact metric space. Then each function f in $\mathcal{B}_1(K)$ can be uniformly approximated by a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous pointwise stabilized functions.*

Proof. $f \in \mathcal{B}_1(K)$, hence there is a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous function pointwise convergent to f. Let us define

$$A_n = \bigcap_{p,q\geq n} \{s \in K : |f_p(s) - f_q(s)| < \varepsilon\} \quad \text{and} \quad B_n = X \setminus A_n$$

Thus A_n is G_δ. It is not difficult to define a Baire-1 function φ_n such that

$$A_n = \{s \in K : \varphi_n(s) = 0\} \quad \text{and} \quad B_n = \{s \in K : \varphi_n(s) > 0\}$$

Now take for each $k \in \mathbb{N}$

$$\psi_{k,n} = \min\{k\varphi_n, 1\}$$

Notice that $(\psi_{k,n})_{k\in\mathbb{N}}$ point-wise converge to $\mathbb{1}_{B_n}$ and it is point-wise stabilized.

To conclude define $g_1 = f_1$ and for each $k \geq 2$

$$g_k = f_1 + (f_2 - f_1)\psi_{1,k} + \ldots + (f_k - f_{k-1})\psi_{k-1,n}$$

Thus $(g_k)_{k\in\mathbb{N}}$ is a sequence of continuous functions point-wise stabilized such that

$$\|f - g\|_\infty < \varepsilon$$

□

Let 2^ω be the Cantor set and let $(K_{r,s})_{r\in\mathbb{N},1\leq s\leq 2^r}$ the system of Cantor intervals, i.e.

$$2^\omega = \bigcap_{r\in\mathbb{N}} \bigcup_{1\leq s\leq 2^r} K_{r,s}.$$

For each $r \in \mathbb{N}$, if f is a real function defined on $K_{r,1}$ and $s \in \{1, \ldots 2^r\}$, we will denote with $\overline{f}$ the periodic extension of f to 2^ω, i.e. the real function defined on $\bigcup_{1\leq s\leq 2^r} K_{r,s}$ such that

$$\overline{f}(t) = f\left(t - \frac{s-1}{2^r}\right) \quad \text{if } t \in K_{r,s}$$

Lemma 4.14. *Consider the sequence $(\varepsilon_r)_{r\in\mathbb{N}}$ with $\varepsilon_r = 1/5^r$, then for all $r \in \mathbb{N}$ we can define inductively A_r and B_r two disjoint G_δ subsets of $K_{r,1}$, a function $\varphi_r \in \mathcal{B}_1(K_{r,1})$, a sequence $(f_{r,n})_{n\in\mathbb{N}}$ in $C(K_{r,1})$ and a subspace X_r of $C(K)$ such that:*

- $\varphi_r(t) = 1$ *for all* $t \in A_r$ *and* $\varphi_r(t) = -1$ *for all* $t \in B_r$ *and* $\|\varphi_r\|_\infty = 1$*;*
- $(f_{r,n})_{n\in\mathbb{N}}$ *is a point-wise stabilized sequence with weak* limit* φ_r*;*
- $\|f_{r,n}\|_\infty \leq 1$ *for all* $n \in \mathbb{N}$*;*
- $X_r := \overline{\text{span}}\{\sum_{s=1}^r \varepsilon_s \overline{f}_{s,n} : n \in \mathbb{N}\}$
- $L(A_r, B_r) > \left[O\left(X_{r-1}, \frac{\varepsilon_r}{8}\right)\right]$

The construction is possible by the following facts:

a. Two disjoint G_δ can be separated by a Baire-1 function, which by Proposition 4.13 is the limit of a stabilizing sequence of continuous functions;

b. Since also the sequences $(\overline{f}_{r,n}, n)_{n\in\mathbb{N}}$ stabilize, by Proposition 4.12, X_r^* are separable. Hence ℓ_1 does not embed in X_r and $O\left(X_r, \delta\right) < \omega_1$ for all $\delta > 0$ by Definition of Bourgain index 3.21.

Theorem 4.15. *For $X = C(2^\omega)$, both $\mathcal{B}_1^0(X)$ and $\mathcal{B}_1^1(X)$ are not vector spaces, and then they cannot be equal to $\mathcal{B}_1(X)$.*

Proof. Let us consider $\delta_r = 2^{-r}$ for each $r \in \mathbb{N}$ and take $\psi_r \in \mathcal{B}_1(K)$, given by

$$\psi_r(t) = \begin{cases} \overline{\varphi}_r(t) & \text{if } t \in K_{r,2^r-1} \\ \psi_r(t) = 0 & \text{otherwise} \end{cases}$$

Let us define $\Phi = \sum_r \varepsilon_r \overline{\varphi}_r$ and $\Psi = \sum_r \delta_r \psi_r$, which are in $\mathcal{B}_1(K)$.

Claim:

- $\Psi \in \mathcal{B}_1^0(X)$ and $\Phi + \Psi \in \mathcal{B}_1^0(X)$
- $\Phi \notin \mathcal{B}_1^1(X)$

Let for all $r, n \in \mathbb{N}$ the function $g_{r,n}$ in $C(K)$ be given by

$$g_{r,n}(t) = \begin{cases} \overline{f}_{r,n}(t) & \text{if } t \in K_{r,2^r-1} \\ 0 & \text{otherwise} \end{cases}$$

Consider the sequence of function $(g_n)_{n\in\mathbb{N}}$ in X, where $g_n = \sum_r \delta_r g_r, n$. Such sequence is bounded by 1 and pointwise stabilizing with limit Ψ. Therefore by Proposition 4.12 $\Gamma = \overline{\operatorname{span}}\{g_n \colon n \in \mathbb{N}\}$ has separable dual. Thus $\Psi \in \mathcal{B}_1^0(X)$.

On the other hand Φ is the pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}}$ in X, where $f_n = \sum_r \varepsilon_r \overline{f}_r, n$ for each $n \in \mathbb{N}$. Thus $\Phi + \Psi$ is the point-wise limit of the sequence $(f_n + g_n)_{n\in\mathbb{N}}$. We will show that $\Lambda = \overline{\operatorname{span}}\{f_n + g_n \colon n \in \mathbb{N}\}$ and Γ are isomorphic and therefore that $\Phi + \Psi \in \mathcal{B}_1^0(X)$.

If $(a_n)_n$ is a finite sequence of scalars, then

$$\left\| \sum_n a_n f_n \right\| \leq \sum_r \varepsilon_r \left\| \sum_n a_n \overline{f}_{r,n} \right\| = \sum_r \varepsilon_r \left\| \sum_n a_n f_{r,n} \right\| = \sum_r \varepsilon_r \left\| \sum_n a_n \varepsilon_{r,n} \right\|$$

$$\leq \frac{4}{5} \sum_r \delta_r \left\| \sum_n a_n \delta_r \varepsilon_{r,n} \right\| \leq \frac{4}{5} \sum_r \delta_r \left\|{}_n a_n g_n \right\| = \frac{4}{5} \left\| \sum_n a_n g_n \right\|$$

and hence

$$\frac{1}{5} \left\| \sum_n a_n g_n \right\| \leqslant \left\| \sum_n a_n (f_n + g_n) \right\| \leqslant \frac{9}{5} \left\| \sum_n a_n g_n \right\|$$

Thus Γ and Λ are isomorphic.

For the second part of the *Claim* we will show that if $(h_m)_{m\in\mathbb{N}}$ is a uniformly bounded sequence in $C(K)$ with pointwise limit Φ, then ℓ_1 embeds in $Y = \overline{\operatorname{span}}\{h_m \colon m \in \mathbb{N}\}$. By the proof of Lemma 4.6 that we may assume $h_m \in \operatorname{co}\{f_n \colon n \geq m\}$ for each $m \in \mathbb{N}$. Thus for all $r \in \mathbb{N}$ we obtain a sequence $(h_{r,m})_m$ in $C(K_{r,1})$ such that for each $m \in \mathbb{N}$:

(i) $h_{r,m}$ are the same convex combinations of the $f_{r,n}(n \geq m)$

(ii) $h_m = \sum_r \varepsilon_r \overline{h}_{r,m}$

We will prove that for each $r \in \mathbb{N}$, there exists some y in Y such that $\|y\| \leq \frac{3}{2}$ and

$$\inf y\left(K_{r,s}\right) < -\frac{1}{8} \quad \text{and} \quad \sup y\left(K_r, s\right) > \frac{1}{8} \quad \text{for all } s = 1, \ldots, 2^r \tag{4.6}$$

Thus $Z = \{z \in Y \colon \|z\| \leq \frac{3}{2}\}$ in not relatively compact in $\mathcal{B}_1(X)$ with the point-wise topology, thus there is $(z_n)_{n\in\mathbb{N}} \in Z$ with no weak Cauchy subsequences and by Theorem 2.24 ℓ_1 embeds in Y. Thus part two of the *Claim* will be completed.

Let us fix $r \in \mathbb{N}$. Since $\lim_m h_{r,m} = \lim_n f_{r,n} = \varphi_r$ point-wise on $K_{r,1}$, it follows from Theorem 3.17 that

$$L\left(A_r, B_r\right) \leq L\left(\varphi_r, -1, 1\right) \leq \left[O\left(C_m, D_m, m\right)\right]$$

where
$C_m = \left\{t \in K_{r,1}; h_{r,m}(t) < -\frac{1}{2}\right\}$ and $D_m = \left\{t \in K_{r,1}; h_{r,m}(t) > \frac{1}{2}\right\}$.
Therefore

$$O\left(X_{r-1}, \frac{\varepsilon_r}{8}\right) < O\left(C_m, D_m, m\right) \tag{4.7}$$

.

By (i), if we define $x_m = \sum_{s=1}^{r-1} \varepsilon_s \overline{h}_{s,m}$ then $(x_m)_{m\in\mathbb{N}}$ is in $B_{X_{r-1}}$.
Let us introduce the tree

$$T = \bigcup_{k=1}^{\infty} \left\{(m_1, \ldots, m_k) \in \mathbb{N}^k \colon m_1 < \ldots < m_k \ \text{ and } \ (x_{m_1}, \ldots, x_{m_k}) \in T\left(x_{r-1}, \frac{\varepsilon_r}{8}\right)\right\}$$

The mapping $m \mapsto x_m$ induces a regular map from T into $T\left(X_{r-1}, \frac{\varepsilon}{8}\right)$. Therefore by (4.7) and by Proposition 3.6

$$o[T] \leq o\left[T\left(X_{r-1}, \frac{\varepsilon_r}{8}\right)\right] < o[T\left(C_m, D_m, m\right)]$$

Hence there are integers $m_1 < \ldots < m_k$ such that $(m_1, \ldots, m_k) \in T\left(C_m, D_m, m\right)$ and $\left\|\sum_{l=1}^k \lambda_l x_{m_l}\right\| < \frac{\varepsilon_r}{8}$ for some $(\lambda_1, \ldots, \lambda_k)$ in $\mathbb{R}^k$ with $\sum_{l\in\mathbb{N}} |\lambda_l| = 1$.

Consider $(\nu_1, \ldots, \nu_k)$ in $\{1, -1\}^k$ so that $\sum_l \nu_l \lambda_l = 1$. Since $(m_1, \ldots, m_k) \in T\left(C_m, D_m; m\right)$ then

$$\bigcap_{l=1}^{k} \nu_l C_{m_l} \neq \emptyset \quad \text{and} \quad \bigcap_{l=1}^{k} \nu_l D_{m_l} \neq \emptyset$$

(with the usual notation $\nu C_m = C_m$ if $\nu = 1$ and $\nu C_m = D_m$ if $\nu = -1$).

Hence, if $u = \sum_l \lambda_l h_{r,m_l}$, then

$$\inf u\,(K_{r,1}) < -\frac{1}{2} \quad \text{and} \quad \sup u\,(K_{r,l}) > \frac{1}{2}$$

and therefore

$$\inf \overline{u}\,(K_{r,s}) < -\frac{1}{2} \quad \text{and} \quad \sup \overline{u}\,(K_{r,s}) > \frac{1}{2}$$

for all $s = 1, \ldots, 2^r$.

Now observe that

$$y = \varepsilon_r^{-1} \sum_l \lambda_l h_{m_l} = \varepsilon_r^{-1} \sum_l \lambda_l x_{m_l} + \overline{u} + \varepsilon_r^{-1} \sum_l \lambda_l \sum_{s>r} \varepsilon_s \overline{h}_{s,m_l}$$

and thus $\|y - \overline{u}\| \leqslant \varepsilon_r^{-1} \left(\frac{\varepsilon_r}{8} + \sum_{s>r} \varepsilon_s \right) = \frac{3}{8}$. Since $\|\overline{u}\| \leq 1$, the element y of Y is such that $\|y\| \leq \frac{3}{8}$ and satisfies (4.6).

$\square$

Chapter 5

Chapter Four Title

5.1 Descriptive set Theory

Descriptive set Theory is the study of subsets of Polish spaces, which are classified inside hierarchies, according to the complexity of their definition and structure.

In the beginning we have the Borel sets of a topological space (X, τ), i.e. the σ-algebra generated by the open sets of X. Their class is denoted by $\mathcal{B}(X)$.

If X is Polish, this class can be further analyzed in a transfinite hierarchy of length ω_1 (the first uncountable ordinal), the **Borel hierarchy**. These classes are denoted by $\mathbf{\Sigma}^0_\xi, \Pi^0_\xi$, for $1 \leq \xi < \omega_1$, where

$$\begin{aligned}
\mathbf{\Sigma}^0_1 = & \text{ open}, \mathbf{\Pi}^0_1 = \text{ closed;} \\
\mathbf{\Sigma}^0_\xi = & \left\{ \bigcup_{n\in\mathbb{N}} A_n : A_n \text{ is in } \mathbf{\Pi}^0_{\xi_n} \text{ for } \xi_n < \xi \right\}; \\
\mathbf{\Pi}^0_\xi = & \text{ the complements of } \mathbf{\Sigma}^0_\xi \text{ sets.}
\end{aligned}$$

(Therefore, $\mathbf{\Sigma}^0_2 = F_\sigma, \mathbf{\Pi}^0_2 = G_\delta, \mathbf{\Sigma}^0_3 = G_{\delta\sigma}, \mathbf{\Pi}^0_3 = F_{\sigma\delta}$, etc.). We have that:

$$\mathcal{B}(X) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi.$$

Beyond the Borel sets one has next the **projective sets**, which are those obtained from the Borel sets by the operations of continuous image and complementation. The class of projective sets ramifies in an infinite hierarchy of length ω (the first infinite ordinal), the **projective hierarchy**, denoted by **P**. We denote by

$$\begin{aligned}
\mathbf{\Sigma}^1_1 = & \text{ analytic}, \mathbf{\Pi}^1_1 = \text{ co-analytic;} \\
\mathbf{\Sigma}^1_{n+1} = & \text{ all continuous images of } \mathbf{\Pi}^1_n \text{ sets;} \\
\mathbf{\Pi}^1_{n+1} = & \text{ the complements of } \mathbf{\Sigma}^1_{n+1} \text{ sets;}
\end{aligned}$$

and
$$\mathbf{P} = \bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}^1_n = \bigcup_{n\in\mathbb{N}} \mathbf{\Pi}^1_n$$

We will now show some important and well known facts in Descriptive Set Theory. We refer the reader the book [Kec95] for the proofs and the details of this section.

Definition 5.1. Let X, Y be two topological spaces. A function $f\colon X \to Y$ is called **Borel** if $f^{-1}(B)$ is in $\mathcal{B}(X)$ for all $B \in \mathcal{B}(Y)$.

Definition 5.2. A subset A of a Polish space X is said to be

- **analytic** if it is the continuous image of a Borel set in a Polish space Y.
- **coanalytic** if $X \setminus A$ is analytic.

We denote by $\Sigma^1_1(X)$ the family of all analytic subsets of X and by $\Pi^1_1(X)$ the family of all coanalytic subsets.

The following result is known as the Lusin separation Theorem.

Theorem 5.3 (Lusin)**.** *Let X be a Polish space and let A, B be two disjoint analytic subset of X. Then there exists $C \in \mathcal{B}(X)$ such that*

1. $A \subseteq C$
2. $B \cap C = \emptyset$

Theorem 5.4 (Souslin)**.** *If X is a Polish space then*
$$\mathcal{B}(X) = \Sigma^1_1(X) \cap \Pi^1_1(X)$$

Proof. ($\subseteq$) Is trivial.
($\supseteq$) Take $A \in \Sigma^1_1(X) \cap \Pi^1_1(X)$, then we can apply Theorem 5.3 to A, A^c, thus there exists $C \in \mathcal{B}(X)$ such that $A \subseteq C$ and $B \cap C = \emptyset$, therefore $A = B$. □

Theorem 5.5. *If F is subspace of the Polish space X, then F endowed with the relative topology is Polish if and only if F is a G_δ subspace of X.*

Let X be a topological space. In this chapter we denote by $\mathcal{F}(X)$ the set of all his closed subsets and by $\mathcal{K}(X)$ the set of all his compact subsets.

It is common to endow $\mathcal{K}(X)$ with the **Vietoris topology**, i.e. the topology generated by the sets
$$\{\mathcal{V}(U_0, \dots, U_n) \colon n \in \mathbb{N}, U_i \text{ open subsets of } Y\}.$$

where

$$\mathcal{V}(U_0, \ldots, U_n) = \{K \in \mathcal{K}(X) \colon K \subseteq U_0 \text{ and } K \cap U_i \neq \emptyset \text{ for all } i \in \{1, \ldots, n\}\}$$

If X is a metric space, it is also possible to define the **Hausdorff Metric** δ_H on $\mathcal{K}(X)$ by

$$\delta_H(H, L) = \max\{\max_{x \in K} d_X(x, L), \max_{y \in L} d(K, y)\}$$

for all $K, L \in \mathcal{K}(X)$.

Remark 5.6. The topology induced by the Hausdorff metric is exactly the Vietoris topology.

Remark 5.7.
- If X is a separable space, so is $\mathcal{K}(X)$.
- If (X, d) is a complete metric space, so is $(\mathcal{K}(X), \delta_H)$.
- Thus if X is a Polish space, so is $\mathcal{K}(X)$.

Proposition 5.8. *If (X, d) is compact metric space, then the Borel sets of $\mathcal{K}(X)$ are generated by*

$$\{\{K \in \mathcal{K}(X) \colon K \cap U \neq \emptyset\} \colon U \text{ open subset of } X\}$$

Definition 5.9. A measure space $(X, \mathcal{S})$, where $\mathcal{S}$ is a σ-algebra of subsets of X, is called **standard Borel** if there is a Polish topology on this set whose Borel σ-algebra coincides with $\mathcal{S}$.

Remark 5.10. Let X be a Polish space. We can construct a compactification of X in the following way:
We take $(x_n)_{n \in \mathbb{N}}$ a dense sequence in X and we define the injective function

$$\begin{aligned} i \colon X &\to [0,1]^\omega \\ x &\mapsto (d(x, x_n))_{n \in \omega} \end{aligned}$$

Then $Y = \overline{i(X)}$ is a metric compactification of X.

Theorem 5.11. *Let X be a Polish space. Then the σ-algebra on $\mathcal{F}(X)$ generated by*

$$\{\{C \in \mathcal{F}(X) \colon F \cap U \neq \emptyset\} \colon U \text{ open subset of } X\}$$

is Standard Borel.
*The space $\mathcal{F}(X)$ with this σ-algebra is called the **Effros Borel space**.*

Proof. Let $\overline{X}$ be a metric compactification of X. Since X is a Polish subspace of the Polish space Y, by Theorem 5.5 X is a G_δ subset of Y. Thus $X = \bigcap_{n\in\mathbb{N}} U_n$ where U_n is an open set of Y for all $n \in \mathbb{N}$. Let

$$Z = \{\overline{F}^Y : F \in \mathcal{F}(X)\} \subseteq \mathcal{K}(Y)$$

and define

$$\begin{aligned} \Phi \colon \mathcal{F}(X) &\to \mathcal{K}(X) \\ F &\mapsto \overline{F}^Y \end{aligned}$$

If $\overline{F_1}^Y = \overline{F_2}^Y$ then $F_1 = X \cap \overline{F_1}^Y = X \cap \overline{F_2}^Y = F_2$ thus Φ is injective.

Claim Z is a G_δ and therefore a Polish space by Remark 5.7.
Indeed $K \in Z \iff K \cap X = K \cap (\bigcap_{n\in\mathbb{N}} U_n)$ is dense in $K \iff K \cap U_n$ is dense in K for all $n \in \mathbb{N}$ by the Baire Category Theorem 2.4. Let $(V_m)_{m\in\mathbb{N}}$ be a basis for Y, then
$K \cap U_n$ is dense in K for all $n \in \mathbb{N} \iff (K \cap V_m \neq \emptyset \implies K \cap (V_m \cap U_m) \neq \emptyset)$ for all $n, m \in \mathbb{N}$. Therefore

$$Z = \bigcap_{n,m,l\in\mathbb{N}} \Big(\mathcal{V}(B(V_m^c, 1/l)) \cup \mathcal{V}(Y, U_n \cap V_m) \Big).$$

Now we ca transfer back the relative topology on Z via Φ, to get a Polish topology on $\mathcal{F}(X)$. We need to verify that the Borel σ-algebra generated by this topology on $\mathcal{F}(X)$ is the Effros-Borel one. Indeed by Proposition 5.8 $\mathcal{B}(\mathcal{K}(Y))$ is generated by

$$\{\{K \in \mathcal{K}(Y) \colon K \cap U \neq \emptyset\} \colon U \text{ open subset of } Y\}$$

thus $\mathcal{B}(\mathcal{F}(X))$ is generated by

$$\begin{aligned} &\{\Phi^{-1}(\{K \in \mathcal{K}(Y) \colon K \cap U \neq \emptyset\}) \colon U \text{ open subset of } Y\} \\ = &\{\{F \in \mathcal{F}(X) \colon \overline{F}^Y \cap U \neq \emptyset\} \colon U \text{ open subset of } Y\} \\ = &\{\{F \in \mathcal{F}(X) \colon (F \cap X) \cap U \neq \emptyset\} \colon U \text{ open subset of } Y\} \\ = &\{\{F \in \mathcal{F}(X) \colon F \cap V \neq \emptyset\} \colon V \text{ open subset of } X\} \end{aligned}$$

□

The following is an important Selection Theorem by Kuratowski and Ryll-Nardzewski, whose proof can be found in [Kec95].

Theorem 5.12. *If X is a Polish space, then there is a sequence of Borel functions $d_n \colon \mathcal{F}(X) \to X$, such that for every non-empty $F \in \mathcal{F}(X)$ $(d_n(F))_{n\in\mathbb{N}}$ is dense in F.*

Lemma 5.13. *If (X, d) is a compact metric space, then X is homeomorphic to a closed subset of $[0,1]^\omega$.*

Proof. We may and do assume that the metric on X is bounded by 1. Since X is a compact metric space, there exists a countable dense subset $\{x_n : n \in \mathbb{N}\}$. We define $F : X \to [0,1]^\omega$ by setting

$$F(x) := (d(x, x_1), d(x, x_2), \ldots, d(x, x_n), \ldots)$$

The coordinate functions $\pi_n \circ F : X \to [0,1]$ are continuous therefore the function F is continuous. We claim that F is one-one. Suppose that $x, y \in X$ are such that $F(x) = F(y)$. Since $\{x_n\}$ is dense in X, there exists a sequence (x_{n_k}) such that $x_{n_k} \to x$ as $k \to \infty$. Hence $d(x_{n_k}, x) \to 0$ as $k \to \infty$. Since $F(x) = F(y)$, it follows that $d(x, x_n) = d(y, x_n)$ for all n. In particular, $d(y, x_{n_k}) = d(x, x_{n_k}) \to 0$. Since the limit of a sequence in a metric space is unique, we deduce that $x = y$. This establishes our claim.

Since X is compact and $[0,1]^\omega$ is Hausdorff, it follows that $F : X \to F(X)$ is a homeomorphism. □

Theorem 5.14. *Every compact metric space is a continuous image of the Cantor set.*

Proof. By Lemma 5.13 we can assume that the given compact metric space X is a subset of $[0,1]^\omega$.

Consider the map $g : 2^\omega \to [0,1]$ given by $g(x) = \sum_k x_k/2^{k+1}$; g is continuous and surjective, thus the unit interval $[0,1]$ is the continuous image of the Cantor set.

The map g induces

$$\begin{aligned} \overline{g} \colon (2^\omega)^\omega &\to [0,1]^\omega \\ (x_n)_{n\in\mathbb{N}} &\mapsto (g(x_n))_{n\in\mathbb{N}} \end{aligned}$$

which is continuous and surjective. Since 2^ω and $(2^\omega)^\omega$ are isomorphic, we then have a continuous function F from the Cantor set 2^ω onto $[0,1]^\omega$. Then $F^{-1}(X)$ is a closed subset of 2^ω and it is mapped by F onto X.

To conclude, notice that every closed subset of the Cantor set is a retract, and therefore it is the continuous image of the Cantor set. □

Theorem 5.15. *Every separable Banach space is isometrically isomorphic to a subspace of $C(2^\omega)$.*

Proof. By hypothesis, (B_{X^*}, w^*) is a compact and complete metric space. Thus by Theorem 5.14 there is a continuous and surjective function $f \colon 2^\omega \to (B_{X^*}, w^*)$.

Define $T \colon X \to C(2^\omega)$ such that

$$T(x)(\sigma) = f(\sigma)(x) \quad \text{for all } \sigma \in C(2^\omega),\ x \in X$$

T is linear, and by surjectivity of f, it is easy to see that T is an isometric isomorphism. □

Remark 5.16. By previous Theorem, we can identify the class $\mathcal{SB}$ of all separable Banach space as a subset of $\mathcal{F}(C(2^\omega))$. If we endow $\mathcal{F}(C(2^\omega))$ with the Effros Borel structure, then $\mathcal{SB}$ is a Borel subspace.

Indeed $F \in \mathcal{F}(C(2^\omega))$ is a vectorial subspace if and only if

$$\forall n, m \in \mathbb{N},\, \forall p, q \in \mathbb{Q}\; [pd_n(F) + qd_m(F)] \in F$$

where (d_n) is a sequence as in 5.12.

The following is a fundamental theorem, whose proof can be found in [[Kec95], Theorem 27.1].

Theorem 5.17. *The set $\mathcal{IF}$ is Σ^1_1-complete.*

5.2 On certain classes of first Baire functions and geometry of Banach spaces

In this chapter is presented the original work [MP22], by the author and his supervisor. The article's aim is to study the two spaces $B^0_1(X)$ and $B^1_1(X)$ introduced by Bourgain, from the descriptive set theory point of view. It is shown that a quantitative version of Bourgain's theorem holds; namely, the family

$$\{X \in \mathcal{SB} \colon B^1_1(X) \subsetneq B_1(X)\}$$

is not Borel in the family of all separable Banach spaces.

For more detail on the proofs of this section, we refer the author to the article [MP22].

Proposition 5.18. *Let X be a separable Banach space, Y be a closed subspace of X and $i : Y \longrightarrow X$ be the natural embedding. If $y^{**} \in B_1(Y)$, then*

$$y^{**} \in B^1_1(Y) \text{ if and only if } i^{**}(y^{**}) \in B^1_1(X).$$

5.2.1 An auxiliary space

Let us denote by $(u_n)_n$ the standard Schauder basis of $C(2^\omega)$ and by $c_{\underline{u}} > 0$ its basis constant. We denote by $c_{00}(T)$ the space of finitely supported function from $T = \omega^{<\omega}$ to $\mathbb{R}$ and by $\chi_s : T \longrightarrow \{0, 1\}$ the characteristic function of $\{s\}$ for every $s \in T$. Thus $c_{00}(T) = \operatorname{span}\{\chi_s \;:\; s \in T\}$.

An *admissible choice of intervals* is a finite set $\{I_j \;:\; 0 \leq j \leq k\}$ of intervals of T such that every branch of T meets at most one of these intervals.

We define the following norm on $c_{00}(T)$:

$$\|y\|_2 = \sup \left[\sum_{j=0}^{k} \left\| \sum_{s \in I_j} y(s)\ u_{|s|} \right\|^2_{C(2^\omega)} \right]^{\frac{1}{2}}$$

where the supremum is taken over $k \in \omega$ and over all admissible choice of intervals $\{I_j \ : \ 0 \leq j \leq k\}$.

We let $U_2(T)$ to be the completion of $c_{00}(T)$ under the norm $\| \cdot \|_2$. In the sequel, for $A \subseteq \omega^{<\omega}$, we denote by $U_2(A)$ the closed subspace of $U_2(T)$ generated by $\{\chi_s \ : \ s \in A\}$.

Lemma 5.19. *Let b be a branch of T. Then*

(i) The space $U_2(b)$ is isomorphic to $C(2^\omega)$.

(ii) If $\theta \in \mathcal{T}$ and if b is a branch of θ, then $U_2(b)$ is complemented in $U_2(\theta)$.

Lemma 5.20. *Let $(A_i)_{i\in\omega}$ be a sequence of subsets of T such that every branch meets at most one of these subsets. Then the spaces*

$$U_2(\bigcup_{i\in\omega} A_i) \text{ and } (\bigoplus_{i\in\omega} U_2(A_i))_{\ell_2} \text{ are isometric}$$

The following is a key result.

Theorem 5.21. *Let $\theta \in \mathcal{T}$.*

(i) If θ is ill founded, then $B_1^1(U_2(\theta)) \subsetneqq B_1(U_2(\theta))$;

(ii) If θ is well founded, then $B_1^1(U_2(\theta)) = B_1(U_2(\theta))$;.

Proof. (i) If θ is ill founded, we pick b a branch of θ. By Lemma 5.19, $U_2(\theta)$ contain a complemented copy of $U(b) \simeq C(2^\omega)$. By Theorem 4.15 and Proposition 5.18, it follows that $B_1^1(U_2(\theta)) \subsetneqq B_1(U_2(\theta))$.

(ii) For $\theta \in \mathcal{T}$, $s \in T$ and $i \in \omega$, we define

$$s^\frown\theta = \{s^\frown t : t \in \theta\}.$$

Since $U_r(\theta) = U_r(\emptyset^\frown\theta)$, to prove the theorem it is enough to show the following *Claim* If θ is well founded, then for any $s \in T$, $U_2(s^\frown\theta)$ is reflexive.

We will show the Claim using transfinite induction on $o(\theta)$.

We assume that for every tree $\tau \in \mathcal{T}$ such that $o(\tau) < \alpha < \omega_1$, $U_2(s^\frown\tau)$ is reflexive for any $s \in T$.

Let $\theta \in \mathcal{T}$ such that $o(\theta) = \alpha$, and for $s \in T$ let $N_s = \{i \in \omega : s^\frown(i) \in \theta\}$. We let $A_i = s^\frown(i)^\frown\theta_i$ for $i \in N_s$, so that $\cup_{i\in N_s} A_i = s^\frown(\theta \setminus \{s\})$ and every branch of T meets at most one of the A_i's. If $i \in N_s$, by Proposition 3.7 we get $o(\theta_i) < \alpha$, thus $U_2(A_i)$ is reflexive by the induction hypothesis. By Lemma 5.20, we have

$$U_2(s^\frown(\theta \setminus \{s\})) = U_2(\bigcup_{i\in N_s} A_i) = (\bigoplus_{i\in N_s} U_2(A_i))_{\ell_2},$$

and thus $U_2(s^\frown(\theta \setminus \{s\}))$ is reflexive.

Since $\{\chi_{s_j} : \ j \in \omega, \ s_j \in s^\frown\theta\}$ is a basis of $U_2(s^\frown\theta)$ with the first element χ_s and the other element generate $U_2(s^\frown(\theta \setminus \{s\}))$. Then, we have that $U_2(s^\frown\theta) \cong \mathbb{R} \times U_2(s^\frown(\theta \setminus \{s\}))$. Therefore $U_2(s^\frown\theta)$ is reflexive. Since reflexive, it easily follows $B_1^1(U_2(\theta)) = B_1(U_2(\theta))$ (which are both equal to $U_2(\theta)$!) □

Similarly, one gets

Theorem 5.22. *Let $\theta \in \mathcal{T}$.*

(i) If θ is ill founded, then $U_2(\theta) \subsetneqq B_1^0(U_2(\theta)) \subsetneqq B_1(U_2(\theta))$;

(ii) If θ is well founded, then $U_2(\theta) = B_1^0(U_2(\theta)) = B_1(U_2(\theta))$;.

5.2.2 Main results

Lemma 5.23. *The map $\varphi : \mathcal{T} \longrightarrow \mathcal{SE}$ defined by*

$$\varphi(\theta) = U_2(\theta)$$

is Borel.

Definition 5.24. A subset A of a Polish space X is said to be Σ_1^1**-hard** or **complete analytic** if for every Polish space Y, any $B \subseteq Y$ analytic can be written as $B = f^{-1}(A)$ for some Borel map $f : Y \to X$.

Notice that, since there are analytic sets that are not Borel, it follows that Σ_1^1-hard sets are not Borel.

We can now prove the main result of the article [MP22].

Theorem 5.25. *The family of all separable Banach spaces X such that $B_1^1(X) \subsetneqq B_1(X)$ is Σ_1^1-hard. In particular, it cannot be Borel in $\mathcal{SB}$.*
Similarly $\{X \in \mathcal{SB}\colon B_0^1(X) \subsetneqq B_1(X)\}$ is Σ_1^1-hard.

Proof. Let us denote by $\mathcal{F}$ such a family. If $\mathcal{F}$ was Borel, then $\mathcal{IF} \subseteq \varphi^{-1}(\mathcal{F})$, and the inclusion has to be strict. Indeed, $\mathcal{IF}$ is Σ_1^1-complete (5.17) and thus it cannot be Borel, while $\varphi^{-1}(\mathcal{F})$ is Borel by assumption. Therefore, there must exists $\theta \in \mathcal{WF}$ such that $\varphi(\theta) \in \mathcal{F}$. This is in contrast with Theorem 5.21(*ii*).

For the second part, just use Theorem 5.22 instead. □

We now place the two classes inside the Projective hierarchy.

Theorem 5.26. *The family of all separable Banach spaces X such that*

$$B_1^1(X) \subsetneq B_1(X)$$

is Σ_3^1.

Proof. Let us denote by $\mathcal{F}$ such a family. In [Bos94] it has been proved that

$$C_{\ell_1} = \{X \in \mathcal{SB} :\ \ell_1 \text{ is isomorphic to a subspace of } X\}$$

is analytic. Observe that a sequence $(x_n)_n \subseteq X \subseteq C(B_{X^*})$ w*-converges to some $x^{**} \in X^{**} \setminus X$ if it is weak-Cauchy. Moreover, a sequence $(x_n)_n$ is weakly Cauchy if and only if given increasing sequence (k_n) and (j_n) of positive integers, then the sequence $(x_{k_n} - x_{j_n})_n$ is weakly null.

Claim $\mathcal{W}(X) = \{(x_n)_n \subseteq X^\omega \text{ weakly cauchy}\}$ is coanalytic.
Denote with $[\omega]$ the set consisting of all increasing sequences of natural number. In [Bra14, Theorem 20] it has been proved that $\mathcal{N}(X) = \{(x_n)_n \subseteq X^\omega \text{ weakly null}\}$ is coanalytic, hence

$$A = \{((x_n)_n, (s_n)_n, (t_n)_n) \subseteq X^\omega \times [\omega]^\omega \times [\omega]^\omega \colon (x_{s_n} - x_{t_n})_n \in \mathcal{N}(X)\}$$

is coanalytic. thus

$$\mathcal{W}(X) = \{(x_n)_n \subseteq X^\omega \colon ((x_n)_n, (s_n)_n, (t_n)_n) \in A\ \forall (s_n)_n, (t_n)_n \in [\omega]^\omega\}$$

stays coanalytic.

Note that for a space $X \in \mathcal{SB}$ the equality $B_1(X) = B_1^1(X)$ holds if for every $(x_n)_n \subseteq X$ which weak* converges to some $x^{**} \in X^{**}$ exists a sequence $(y_n)_n \subseteq X$ which weak* converges to the same $x^{**} \in X^{**}$ such that $\overline{\text{span}}\{y_n : n \in \omega\}$ does not contain a copy of ℓ_1. Since the map $G_X : X^\omega \longrightarrow \mathcal{SB}$, $(x_n)_n \longmapsto \overline{\text{span}}\{x_n : n \in \omega\}$ is Borel,

$$B = \{(X, (x_n)_n, (y_n)_n) \subseteq \mathcal{SB} \times X^\omega \times X^\omega \colon (x_n)_n, (y_n)_n \subseteq X,\ (x_n)_n, (y_n)_n \in \mathcal{W}(X),$$

$$w^* - \lim(x_n)_n = w^* - \lim(y_n)_n,\ (y_n)_n \in [G_X^{-1}(C_{\ell_1})]^C\}$$

is coanalytic, thus

$$C = \{(X, x_n)_n \colon \exists (y_n)_n \in X^\omega \text{ such that} (X, (x_n)_n, (y_n)_n) \in B\}$$

is Σ_2^1, hence

$$\mathcal{G} = \{X \in \mathcal{SB} \colon (X, (x_n)_n) \in C\ \forall (x_n)_n \in X^\omega\}$$

is Π_3^1.

Observe that $\mathcal{G} = \{X \in \mathcal{SB} \colon B_1^1(X) = B_1(X)\}$ consequently $\mathcal{F} = \mathcal{G}^C$ is Σ_3^1. $\square$

Similarly we get

Theorem 5.27. *The family of all separable Banach spaces X such that*

$$B_0^1(X) \subsetneqq B_1(X)$$

is Σ_3^1.

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